18.175: Lecture 3

Random variables and distributions

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Characterizing measures on $\mathbb{R}^d$

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Recall definitions

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0,1]$ is the probability function.

- **σ-algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.

- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.

- Measure μ is **probability measure** if $\mu(\Omega) = 1$.

- The **Borel σ-algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

Recall σ-algebra story

- Want, a priori, to define measure of *any* subsets of $[0, 1)$.

- Find that if we allow the axiom of choice and require measures to be countably additive (as we do) then we run into trouble. No valid translation invariant way to assign a finite measure to all subsets of $[0, 1)$.

- Could toss out the axiom of choice... but we don't want to. Instead we only define measure for certain "measurable sets". We construct a σ-algebra of measurable sets and let probability measure be function from σ-algebra to $[0, 1]$.

- Borel σ-algebra is generated by open sets. Sometimes consider "completion" formed by tossing in measure zero sets.

- Caratheéodory Extension Theorem tells us that if we want to construct a measure on a σ-algebra, it is enough to construct the measure on an algebra that generates it.

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Recall construction of measures on $\mathbb{R}$

- Write $F(a) = P\big((-\infty, a]\big)$.
- **Theorem:** for each right continuous, non-decreasing function F, tending to 0 at $-\infty$ and to 1 at ∞, there is a unique measure defined on the Borel sets of $\mathbb{R}$ with $P((a, b]) = F(b) - F(a)$.
- Proved using Caratheéodory Extension Theorem.

- Want to have $F(x) = \mu(-\infty, x_1] \times (\infty, x_2] \times \ldots \times (-\infty, x_n]$.
- Given such an F, can compute μ of any finite rectangle of form $\prod(a_i, b_i]$ by taking differences of F applied to vertices.
- **Theorem:** Given F, there is a unique measure whose values on finite rectangles are determined this way (provided that F is non-decreasing, right continuous, and assigns a non-negative value to each rectangle).
- Also proved using Caratheéodory Extension Theorem.

Characterizing measures on $\mathbb{R}^d$

Random variables

Defining random variables

- Random variable is a *measurable* function from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$. That is, a function $X : \Omega \to \mathbb{R}$ such that the preimage of every set in $\mathcal{B}$ is in $\mathcal{F}$. Say X is $\mathcal{F}$-**measurable**.

- Question: to prove X is measurable, is it enough to show that the pre-image of every open set is in $\mathcal{F}$?

- **Theorem:** If $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{A}$ and $\mathcal{A}$ generates $\mathcal{S}$, then X is a measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$.

- Example of random variable: indicator function of a set. Or sum of finitely many indicator functions of sets.

- Let $F(x) = F_X(x) = P(X \leq x)$ be **distribution function** for X. Write $f = f_X = F_X'$ for **density function** of X.

- What functions can be distributions of random variables?

- Non-decreasing, right-continuous, with $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.

Examples of possible random variable laws

- Other examples of distribution functions: uniform on $[0, 1]$, exponential with rate λ, standard normal, Cantor set measure.
- Can also define distribution functions for random variables that are a.s. integers (like Poisson or geometric or binomial random variables, say). How about for a ratio of two independent Poisson random variables? (This is a random rational with a dense support on $[0, \infty)$.)
- Higher dimensional density functions analogously defined.

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