18.175: Lecture 23

Random walks

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Outline

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Random walks

Stopping times

Arcsin law, other SRW stories

Exchangeable events

- Start with measure space $(S, \mathcal{S}, \mu)$. Let $\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}$, let $\mathcal{F}$ be product σ-algebra and P the product probability measure.
- **Finite permutation** of $\mathbb{N}$ is one-to-one map from $\mathbb{N}$ to itself that fixes all but finitely many points.
- Event $A \in \mathcal{F}$ is permutable if it is invariant under any finite permutation of the ω_i.
- Let $\mathcal{E}$ be the σ-field of permutable events.
- This is related to the tail σ-algebra we introduced earlier in the course. Bigger or smaller?

Hewitt-Savage 0-1 law

- If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{A}$ then $P(A) \in \{0, 1\}$.
- **Idea of proof:** Try to show A is independent of itself, i.e., that $P(A) = P(A \cap A) = P(A)P(A)$. Start with measure theoretic fact that we can approximate A by a set A_n in σ-algebra generated by $X_1, \ldots X_n$, so that symmetric difference of A and A_n has very small probability. Note that A_n is independent of event A_n' that A_n holds when $X_1, \ldots, X_n$ and $X_{n_1}, \ldots, X_{2n}$ are swapped. Symmetric difference between A and A_n' is also small, so A is independent of itself up to this small error. Then make error arbitrarily small.

Application of Hewitt-Savage:

- If X_i are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^n X_i$ is a **random walk** on $\mathbb{R}^n$.
- **Theorem:** if S_n is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all n
  - $S_n \to \infty$
  - $S_n \to -\infty$
  - $-\infty = \liminf S_n < \limsup S_n = \infty$
- **Idea of proof:** Hewitt-Savage implies the $\limsup S_n$ and $\liminf S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if X_1 is not a.s. constant, then both values would depend on X_1 if they were not in $\pm\infty$

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Stopping time definition

- Say that T is a **stopping time** if the event that $T = n$ is in $\mathcal{F}_n$ for $i \leq n$.
- In finance applications, T might be the time one sells a stock. Then this states that the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

Stopping time examples

- Let $A_1, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$.
- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
  6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.
- Answer: first four, not last two.

Stopping time theorems

- **Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. and N a stopping time with $N < \infty$.
- Conditioned on stopping time $N < \infty$, conditional law of $\{X_{N+n}, n \geq 1\}$ is independent of $\mathcal{F}_n$ and has same law as original sequence.
- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.
- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?
- What is $\mathbb{E}N$?

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Reflection principle

- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

- Suppose that in election candidate A gets α votes and B gets $\beta < \alpha$ votes. What's probability that A is a head throughout the counting?
- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.
- Theorem for amount of positive positive time.

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