18.175: Lecture 10

Zero-one laws and maximal inequalities

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Outline

Recollections

Kolmogorov zero-one law and three-series theorem

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Kolmogorov zero-one law and three-series theorem

Recall Borel-Cantelli lemmas

- **First Borel-Cantelli lemma:** If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \ \text{i.o.}) = 0$.
- **Second Borel-Cantelli lemma:** If A_n are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$.

Recall strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \dots$ are i.i.d. real-valued random variables with expectation m and $A_n := n^{-1} \sum_{i=1}^{n} X_i$ are the *empirical means* then $\lim_{n\to\infty} A_n = m$ almost surely.

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Kolmogorov zero-one law

- Consider sequence of random variables X_n on some probability space. Write $\mathcal{F}'_n = \sigma(X_n, X_{n_1}, \ldots)$ and $\mathcal{T} = \cap_n \mathcal{F}'_n$.
- $\mathcal{T}$ is called the **tail σ-algebra**. It contains the information you can observe by looking only at stuff arbitrarily far into the future. Intuitively, membership in tail event doesn't change when finitely many X_n are changed.
- Event that X_n converge to a limit is example of a tail event. Other examples?
- **Theorem:** If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $P(A) \in \{0, 1\}$.

Kolmogorov zero-one law proof idea

- **Theorem:** If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $P(A) \in \{0, 1\}$.
- **Main idea of proof:** Statement is equivalent to saying that A is independent of itself, i.e., $P(A) = P(A \cap A) = P(A)^2$. How do we prove that?
- Recall theorem that if $\mathcal{A}_i$ are independent π-systems, then σA_i are independent.
- Deduce that $\sigma(X_1, X_2, \ldots, X_n)$ and $\sigma(X_{n+1}, X_{n+1}, \ldots)$ are independent. Then deduce that $\sigma(X_1, X_2, \ldots)$ and $\mathcal{T}$ are independent, using fact that $\cup_k \sigma(X_1, \ldots, X_k)$ and $\mathcal{T}$ are π-systems.

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