Lecture 12

March 30^{th}, 2004

$\mathcal{R}$**emark.** Since many of our results rely on the regularity of the Newtonian Potential, and hence use Proposition 2 of Lecture 9, we will assume througout that the Hölder constant α ranges in the *open* interval $(0, 1)$.

Review from last time

$\mathcal{R}$***egularity Theorem.*** *$B \subseteq \mathbb{R}^n$, a ball, $u \in C^2(B) \cap C^0(\bar{B}), f \in C^\alpha(\bar{B})$, with $0 < \alpha < 1$. Suppose u solves Laplace's Equation: $\Delta u = f$ on B, $u = 0$ on ∂B. Then $u \in \mathcal{C}^{2,\alpha}(\bar{B})$.*

In the interior of B, just use estimates on the Newtonian Potential (NP) and on harmonic functions. On the boundary of B use translation & inversion maps to map ball to upper half plane with flat boundary. Then note that the estimates on the NP work upto the boundary and an inversion map is smooth away from the origin. ■

$\mathcal{C}$***orollary.*** *$\varphi \in \mathcal{C}^{2,\alpha}(\bar{B}), f \in \mathcal{C}^\alpha(\bar{B})$, with $0 < \alpha < 1$.. Then Poisson's Equation: $\Delta u = f$ on B, $u = \varphi$ on ∂B, has a unique solution $u \in \mathcal{C}^{2,\alpha}(\bar{B})$.*

By the above *if* we can solve for v such that $\Delta v = f - \Delta\varphi$ on B, $v = 0$ on ∂B, then $v \in \mathcal{C}^{2,\alpha}(\bar{B})$. Let $u := v + \varphi \in \mathcal{C}^{2,\alpha}(\bar{B})$. This u solves our original equation! So we just need to be able to *solve uniquely* the above homogeneous equation with a $\mathcal{C}^2(B) \cap \mathcal{C}^0(\bar{B})$ solution. Then the Theorem will guarantee it is actually $\mathcal{C}^{2,\alpha}(\bar{B})$.

In order to do that, set $w := \mathrm{NP}(g)$, where $g := f - \Delta\varphi \in \mathcal{C}^\alpha$ (as $f \in \mathcal{C}^\alpha, \varphi \in \mathcal{C}^{2,\alpha}$). Indeed $w \in \mathcal{C}^2(B) \cap \mathcal{C}^0(\bar{B})$ from the elementary properties of the Newtonian Potential. Furthermore $\Delta w = g$. If we could make sure somehow the boundary values would be 0 we would be done as all assumptions of the Theorem would hold. In order to do that, we need to find a function

$h \in \mathcal{C}^2(B) \cap \mathcal{C}^0(\bar{B})$ solving $\begin{cases} \Delta h = 0 & \text{on } B, \\ h = -w & \text{on } \partial B \end{cases}$. And indeed by Poisson's Integral Formula we can do this. Letting $v := w + h \in \mathcal{C}^2(B) \cap \mathcal{C}^0(\bar{B})$ we have indeed a the required solution for the homogeneous problem. $\blacksquare$

Solving Poisson's/Laplace's equation with regularity upto the boundary on general domains

$\mathcal{S}$uppose we are given an (open) domain $\Omega \subseteq \mathbb{R}^n$, different than a ball, or equivalently some open subset in a Riemannian manifold (M, g), and that we would like to develope a similar theory for the Poisson and Laplace equations on these domains. In other words prove a priori estimates upto the boundary for these domains.

Localizing to a neighborhood in $\mathbb{R}^n$ of a point on the boundary $\partial\Omega$ intersected with $\bar{\Omega}$, we could map it to a neighborhood of $\overline{\mathbb{H}} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \big| x_n \geq 0\}$. This localization is tantamount to working with the manifolds local coordinates, and then we must work with Δ_g, the Riemannian Laplacian.

We see that indeed we will be able to extend our theory to these generalized domains once we show our boundary estimates hold for general elliptic operators.

Constant coefficients operators

Let $L_0 u(x) = A^{ij} \mathrm{D}_{ij} u(x) = f(x)$ with A^{ij} a constant matrix satisfying $0 < \lambda |v|^2 \leq A^{ij} v_i v_j \leq \Lambda |v|^2, \forall\, 0 \neq v \in \mathbb{R}^n$. This two-sided inequality will be referred to as *uniform ellipticity.*

Theorem. *Let u be as above and $0 < \alpha < 1$.*

I. If $u \in \mathcal{C}^2(\Omega)$, $f \in \mathcal{C}^\alpha(\Omega)$, then $\forall\ \Omega' \Subset \Omega$ (i.e "Ω' precompact in Ω") there exists $C = C(\lambda, \Lambda, \Omega', \Omega, n)$ such that

$$||u||_{\mathcal{C}^{2,\alpha}(\Omega')} \leq C \cdot (||u||_{\mathcal{C}^0(\Omega)} + ||f||_{\mathcal{C}^\alpha(\Omega)}).$$

II. If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\Omega \cup T)$, $f \in \mathcal{C}^\alpha(\Omega \cup T)$ *and* $u = 0$ *on* T, *then* $\forall\ \Omega' \Subset \Omega$ *there exists* $C = C(\lambda, \Lambda, \Omega', \Omega, n)$ *such that*

$$||u||_{\mathcal{C}^{2,\alpha}(\Omega' \cup T')} \leq C \cdot (||u||_{\mathcal{C}^0(\Omega \cup T)} + ||f||_{\mathcal{C}^\alpha(\Omega \cup T)}),$$

where $T' := \Omega' \cap T$. *We assume that* T *is a flat boundary portion (portion of a hyperplane in* $\mathbb{R}^n$*) contained in* $\partial\Omega$ *.*

Setup: Let H be an invertible linear transformation represented by multiplication by a constant matrix H_{kl}, and let H^{-1} denote its inverse. Being linear, by rotating if necessary, we may assume it maps the upper half space to itself, and that the flat boundary portion remains flat. Put $\tilde{u} := u \circ H^{-1}$ and $y = Hx$. Then $\tilde{u} : \Omega \longrightarrow \mathbb{R}$,

$$\mathrm{D}_i \tilde{u}(y) = \mathrm{D}_l u(H^{-1}y) \cdot H^{-1}_{li} \text{ (summation)}$$

from D applied to $u\left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \cdot [H^{-1}] \right) = u\big([\, y_l H^{-1}_{l1}, \ldots, H^{-1}_{ln} y_n \,] \big)$. Then

$$\mathrm{D}_i \mathrm{D}_j \tilde{u}(y) = \mathrm{D}_k \mathrm{D}_l u(H^{-1}y) H^{-1}_{kj} H^{-1}_{li} = (H^{-1})^T \cdot \mathrm{D}^2 u(H^{-1}y) \cdot H^{-1},$$

$$\Rightarrow H^T \cdot \mathrm{D}^2 \tilde{u}(y) \cdot H = \mathrm{D}^2 u(x).$$

Plugging this into our elliptic equation we get $A^{lk} H_{il} \mathrm{D}_i \mathrm{D}_j \tilde{u} H_{jk} = A^{lk} \mathrm{D}_{lk} u(x) = f(x)$, or $H_{il} A^{lk} H_{jk} \mathrm{D}^2 \tilde{u} = (HAH^T) \mathrm{D}^2 \tilde{u}(y) = f(x) = f(H^{-1}y) =: \tilde{f}(y)$.

Choosing appropriate H can diagonalize A: $HAH^T = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Set $P := H\mathrm{diag}(\lambda_1^{-\frac{1}{2}}, \ldots, \lambda_n^{-\frac{1}{2}})$. Then $PAP^T = I$, and in the domain $H(\Omega)$, which has flat boundary, we get the simple Poisson equation $\Delta \tilde{u} = \tilde{f} \in \mathcal{C}^\alpha$. By the theory we developed earlier in the course for this equation on such domains, $\forall\ \tilde{\Omega}' \Subset H(\Omega)$ we have the *interior* estimates

$$||\tilde{u}||_{\mathcal{C}^{2,\alpha}(\tilde{\Omega}')} \leq C \cdot (||\tilde{u}||_{\mathcal{C}^0(H(\Omega))} + ||\tilde{f}||_{\mathcal{C}^\alpha(H(\Omega))}).$$

Now $||u||_{\mathcal{C}^{2,\alpha}(\Omega')} \leq C \cdot ||\tilde{u}||_{\mathcal{C}^{2,\alpha}(H(\Omega'))}$, $||u||_{\mathcal{C}^\alpha(\Omega')} \leq C \cdot ||\tilde{u}||_{\mathcal{C}^\alpha(H(\Omega'))}$, where we have used for the last two the identities $||\tilde{g}||_{\mathcal{C}^0(H(\Omega))} = \sup_{y \in H(\Omega)} |\tilde{g}(y)| = \sup_{x \in \Omega} |g(x)| = ||g||_{\mathcal{C}^0(\Omega)}$ and

$$||\tilde{g}||_{\mathcal{C}^{\alpha}(H(\Omega))} = \sup_{y_1 \neq y_2 \in H(\Omega)} \frac{|\tilde{g}(y_1) - \tilde{g}(y_2)|}{|y_1 - y_2|^{\alpha}} = \sup_{x_1 \neq x_2 \in \Omega} \frac{|g(x_1) - g(x_2)|}{|Hx_1 - Hx_2|^{\alpha}}$$

$$= \sup_{x_1 \neq x_2 \in \Omega} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|^{\alpha}} \Big(\frac{|x_1 - x_2|}{|Hx_1 - Hx_2|} \Big)^{\alpha} \leq ||g||_{\mathcal{C}^{\alpha}(\Omega)} \cdot (\text{smallest eigenvalue of } H)^{-1}.$$

Here we use H is a diffeomorphism. The $\mathcal{C}^{2,\alpha}$ inequality follows similarly using $H^T \mathrm{D}^2 \tilde{u}(y) H = \mathrm{D}^2 u(x)$. Note that since H, H^{-1} are both strictly positive, the above inequalities can be shown to hold in both directions (with different constants). That is to say all norms of $\tilde{u}$ are equivalent to those of u. This observation combined with the above interior estimates for $\tilde{u}$ gives us interior estimates for u in Ω'.

As for boundary estimates (part II of the Theorem): we have seen that we can assume WLOG that H maps the upper half plane to itself. Then our above inequalities for equivalence of the norms extend to the boundary of course, and since our theory (Lecture 11) gives boundary estimates for $\tilde{u}$ we are done. ■

Interpolation

Theorem. *Let $\Omega' \Subset \Omega, u \in \mathcal{C}^{2,\alpha}(\Omega)$, with $0 < \alpha < 1$. For any $\epsilon > 0$, $\exists\, C(\epsilon)$ such that*

$$|u|_{C^{k,\beta}(\Omega')} \leq C(\epsilon) \cdot |u|_{C^{o}(\Omega)} + \epsilon \cdot |u|_{C^{2,\alpha}(\Omega')}.$$

Note these are the semi-norms not the full norms! Also, as $\epsilon \to 0, C(\epsilon) \to \infty$.

The case $k = 1, \beta = 0$. Let $\bar{x}, x' \in \Omega', x'' \in \Omega \setminus \Omega'$, such that all three points lie on a single line segment parallel to the x^i axis and such that $\mathrm{D}_i u(\bar{x}) = \frac{u(x'') - u(x')}{2\epsilon} \leq \frac{2|u|_{C^0(\Omega)}}{2\epsilon}$. From the fact that x'' is not in Ω' we will get a global $\mathcal{C}^0$ norm involved (i.e norm over all Ω instead of just over Ω'). Now let $x \in \Omega'$,

$$\int_x^{\bar{x}} \mathrm{D}_{ii}u = \mathrm{D}_i(\bar{x}) - \mathrm{D}_i(x),$$

from which follows

$$|\mathrm{D}_i(x)| \leq |\mathrm{D}_i(\bar{x})| + |\int_x^{\bar{x}} \mathrm{D}_{ii}u| \leq \frac{1}{\epsilon}|u|_{C^0(\Omega)} + \max_{\substack{x \in \text{segment } \overline{x\bar{x}} \\ \text{in } x^i \text{ direction}}} \mathrm{D}_{ii}u \cdot |x - \bar{x}| \leq \frac{1}{\epsilon}|u|_{C^0(\Omega)} + \epsilon \cdot |u|_{C^2(\Omega')}.$$

The case $k = 2, \beta = 0$**.** Fix i and look at $\mathrm{D}_i u$. Again choose points on a segment in the x^l direction such that $\mathrm{D}_{li}u(\bar{x}) = \dfrac{\mathrm{D}_i u(x'') - \mathrm{D}_i u(x')}{2\epsilon} \leq \dfrac{2|\mathrm{D}u|_{C^0(\text{segment})}}{2\epsilon}$. Now

$$|\mathrm{D}_{li}u(x)| \leq |\mathrm{D}_{li}u(\bar{x})| + |\mathrm{D}_{li}u(x) - \mathrm{D}_{li}u(\bar{x})| \leq \frac{1}{\epsilon}|\mathrm{D}u|_{C^0(\text{segment})} + |\mathrm{D}^2 u|_{\mathcal{C}^2(\Omega)} \cdot |x - \bar{x}|^{\alpha},$$

which by the first case is

$$\leq \frac{1}{\epsilon} \cdot \left(\frac{1}{\epsilon'}|u|_{C^0(\Omega)} + \epsilon' \cdot |u|_{C^2(\Omega)}\right) + \epsilon^a \cdot |u|_{C^{2,\alpha}(\Omega)},$$

hence

$$|u|_{C^2(\Omega')} \leq C \cdot |u|_{C^0(\Omega)} + C' \cdot \epsilon^a \cdot |u|_{C^{2,\alpha}(\Omega')}.$$

For the cases to follow let $x, y \in \Omega'$ and denote by $\bar{x}$ a point (to be chosen later) on the line segment $\overline{xy}$.

The case $k = 0, \beta \in (0, 1]$**.** We note that we can bound $|u|_{C^{0,\beta}(\Omega')}$ in a simple manner since

$$\frac{u(x) - u(y)}{|x - y|^{\beta}} \leq \begin{cases} \dfrac{\mathrm{D}u(\bar{x}) \cdot |x - y|}{|x - y|^{\beta}} \leq |u|_{C^1} \cdot |x - y|^{1-\beta} \leq |u|_{C^1} \cdot \epsilon^{1-\beta}, \text{ if } |x - y| \leq \epsilon, \\ \dfrac{2|u|_{C^0}}{\epsilon^{\beta}}, \text{ if } |x - y| > \epsilon. \end{cases}$$

or in other words

$$|u|_{C^{0,\beta}(\Omega')} \leq \frac{2|u|_{C^0}}{\epsilon^{\beta}} \cdot |u|_{C^0(\Omega)} + \epsilon^{1-\beta} \cdot |u|_{C^1(\Omega')}.$$

The case $k = 1, \beta \in (0, 1]$**.** We again note the dichotomy ($\bar{x}$ as above)

$$\frac{\mathrm{D}_i u(x) - \mathrm{D}_i u(y)}{|x-y|^\beta} \leq \begin{cases} \dfrac{\mathrm{D}\mathrm{D}_i u(\bar{x}) \cdot |x-y|}{|x-y|^\beta} \leq |u|_{C^2} \cdot |x-y|^{1-\beta} \leq |u|_{C^2} \cdot \epsilon^{1-\beta}, \text{ if } |x-y| \leq \epsilon, \\ \dfrac{2|u|_{C^1}}{\epsilon^\beta}, \text{ if } |x-y| > \epsilon. \end{cases}$$

or

$$|u|_{C^{1,\beta}(\Omega')} \leq \frac{2|u|_{C^1}}{\epsilon^\beta} \cdot |u|_{C^1(\Omega)} + \epsilon^{1-\beta} \cdot |u|_{C^2(\Omega)}. \leq C \cdot |u|_{C^0(\Omega)} + C' \cdot |u|_{C^2(\Omega')}$$

where in the last inequality we used one of the previous cases.

The case $k = 2, \beta \in (0, \alpha)$**.** Once again

$$\frac{\mathrm{D}_{ij} u(x) - \mathrm{D}_{ij} u(y)}{|x-y|^\beta} = \frac{\mathrm{D}_{ij} u(x) - \mathrm{D}_{ij} u(y)}{|x-y|^\alpha} \cdot |x-y|^{\alpha-\beta} \leq \begin{cases} |u|_{C^{2,\alpha}(\Omega')} \epsilon^{\alpha-\beta}, \text{ if } |x-y| \leq \epsilon, \\ \dfrac{2|u|_{C^2(\Omega)}}{\epsilon^\beta}, \text{ if } |x-y| > \epsilon. \end{cases}$$

or

$$|u|_{C^{2,\beta}(\Omega')} \leq \frac{2}{\epsilon^\beta} \cdot |u|_{C^2(\Omega')} + \epsilon^{\alpha-\beta} \cdot |u|_{C^{2,\alpha}(\Omega')}. \leq C \cdot |u|_{C^0(\Omega)} + C' \cdot |u|_{C^2(\alpha)} \Omega'$$

where in the last inequality we used one of the previous cases. ■

$\mathcal{R}$**emark.** The Interpolation technique works also for $\Omega' \Subset \Omega$ with flat boundary involved: we get inequalities with the flat boundary portion included, by the theory developed in Lecture 11.