18.156 Lecture Notes
Lecture 30

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In the past few lectures we focused on solutions to the linear homogeneous Schrödinger equation:

$$\partial_t u = i\Delta u \tag{1}$$

with $u \in C^2(\mathbb{R}^d \times \mathbb{R})$. If u satisfies the initial condition $u(x,0) = u_0(x)$ (for sufficiently well-behaved u_0), the solution u is given for all times $t \in \mathbb{R}$ by

$$u(x,t) = e^{it\Delta}u_0 := \left(e^{it(2\pi i\omega)^2}\hat{u}_0\right)^{\vee}.$$

We have developed a variety of bounds for such solutions, which include:

(A) $\left\|e^{it\Delta}u_0\right\|_{L^2_x} = \|u_0\|_{L^2_x}$ for all $t \in \mathbb{R}$.

(B) $\left\|e^{it\Delta}u_0\right\|_{L^\infty_x} \lesssim |t|^{-d/2}\|u_0\|_{L^1_x}$ for all $t \neq 0$.

(C) Interpolation between (A) and (B).

The goal of this lecture is a proof of the following theorem via a combination of (A), (B), and (C):

Theorem 1 ((Strichartz))**.** *If u solves* (1) *and $u(x,0) = u_0(x)$,*

$$\left\|e^{it\Delta}u_0\right\|_{L^\sigma_{x,t}} \lesssim \|u_0\|_{L^2_x}, \tag{2}$$

where $\sigma = \frac{2(d+2)}{d}$.

To understand the strengths and weaknesses of the bounds we already have, let use consider a few specific cases.

Case 1. *$u_0 \in C_c^\infty(B^d(1))$ with $0 \le u_0 \le 1$ and $u_0 = 1$ on $B^d\left(\frac{1}{2}\right)$.*

We first note that $\|u_0\|_{L^p_x} \sim 1$ for all $p \in [1,\infty]$. We have shown in previous lectures that qualitiatively $e^{it\Delta}u_0$ will spread out as t evolves forward. For $t \ge 1$, $|u(x,t)| \sim t^{-d/2}$ for $|x| \le t$, and $u(x,t)$ decays rapidly for $|x| \ge t$.

(A) shows that $\left\|e^{it\Delta}u_0\right\|_{L^2_x} \sim 1$ for all $t \in \mathbb{R}$, which is sharp (it *must* be sharp, since it's an equality).

(B) shows that $\left\|e^{it\Delta}u_0\right\|_{L^\infty_x} \lesssim \left|t^{-d/2}\right|$ for all $t \neq 0$. This is sharp for $|t| \geq 1$, but is pretty useless when $|t| < 1$.

(C) has the same effectiveness as (B).

To expand on this example, consider a slightly more general case:

Case 2. *$u_0 \in C_c^\infty(B^d(R))$ with $R > 0$, $0 \leq u_0 \leq 1$, and $u_0 = 1$ on $B^d\left(\frac{R}{2}\right)$.*

Now (B) and (C) work well for $|t| \geq R^2$, but are weak for $t \in (-R^2, R^2)$. This weakness stems from "focusing." Any L^∞ bound on u must account for the possibility that u is focusing, so that u concentrates in a small region with large values at some future time. We have studied such cases before; a standard example is $w_0 = e^{-iR\Delta}u_0$ with the u_0 from Case 1. Then $|w_0| \sim R^{-d/2}$ on $B^d(R)$, while $\left|e^{iR\Delta}w_0\right| \sim 1$ on $B^d(1)$. In this situation (B) is sharp for $t \sim R$. The focusing with $\left|e^{it\Delta}w_0\right| \sim 1$ only occurs over a small time interval, say for $t \in [-1, 1]$. However, our application of (B) does not prevent focusing from happening over an extended period of time, for instance for all $t \in [R, 2R]$.

Such a "long focus" is precisely the sort of behavior disallowed by Theorem 1. The L^σ bound on space *and time* may permit a focus during a small subset of times, but not over a large time interval. In fact, (B) already controls the length of the focus in Case 2. If we suppose that $e^{iR\Delta}w_0$ is concentrated in $B^d(1)$, we may use $e^{iR\Delta}w_0$ as initial data in Case 1 to show that $e^{it\Delta}w_0$ will not remain focused when $|t - R| \gtrsim 1$. In other words, we may obtain more information about solutions to (1) by using $e^{it\Delta}u_0$ as initial data in (B) and concluding a bound about $e^{is\Delta}u_0$ for $s \neq t$. The proof of Theorem 1 applies (B) to all such pairs (t, s).

We first recall the L^2-unitarity of $e^{it\Delta}$:

Lemma 2. $\left\langle e^{it\Delta}f, g\right\rangle_{\mathbb{R}^d} = \left\langle f, e^{-it\Delta}g\right\rangle_{\mathbb{R}^d}$.

Proof. By the definition of the L^2-inner product on $\mathbb{R}^d$ and Plancherel's theorem:

$$\left\langle e^{it\Delta}f, g\right\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} e^{it\Delta}f\,\bar{g} = \int_{\mathbb{R}^d} e^{it(2\pi i\omega)^2}\hat{f}\,\bar{\hat{g}} = \int_{\mathbb{R}^d} \hat{f}\,\overline{e^{-it(2\pi i\omega)^2}\hat{g}} = \int_{\mathbb{R}^d} f\,\overline{e^{-it\Delta}g} = \left\langle f, e^{-it\Delta}g\right\rangle_{\mathbb{R}^d}.$$

□

With this unitarity we may now proceed with the proof of Strichartz:

Proof (Thoremm 1). By duality,

$$\left\|e^{it\Delta}u_0\right\|_{L^\sigma_{x,t}} = \sup_{\|F\|_{L^{\sigma'}_{x,t}}=1} \int_{\mathbb{R}^d\times\mathbb{R}} e^{it\Delta}u_0\bar{F},$$

where σ' is the dual exponent of σ satisfying $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. By Lemma 2,

$$\sup_{\|F\|_{L^{\sigma'}_{x,t}}=1} \int_{\mathbb{R}^d\times\mathbb{R}} e^{it\Delta}u_0\bar{F} = \sup \int_{\mathbb{R}} \left\langle e^{it\Delta}u_0, F_t\right\rangle\,dt = \sup \int_{\mathbb{R}} \left\langle u_0, e^{-it\Delta}F_t\right\rangle\,dt = \sup \left\langle u_0, \int e^{-it\Delta}F_t\,dt\right\rangle.$$

Hence by Cauchy-Schwarz:

$$\left\|e^{it\Delta}u_0\right\|_{L^\sigma_{x,t}} \leq \|u_0\|_{L^2_x} \sup \left\|\int e^{-it\Delta}F_t\right\|_{F^2_x}.$$

It therefore suffices to check that

$$\sup_{\|F\|_{L^{\sigma'}_{x,t}}=1} \left\| \int e^{-it\Delta} F_t \right\|_{F^2_x} \lesssim 1.$$

We prove this separately as its own lemma:

Lemma 3.

$$\left\| \int e^{-it\Delta} F_t \right\|_{F^2_x} \lesssim \|F\|_{L^{\sigma'}_{x,t}}.$$

Proof.

$$\left\| \int e^{-it\Delta} F_t \right\|^2_{L^2_x} = \left\langle \int_{\mathbb{R}} e^{-it\Delta} F_t \, dt, \int_{\mathbb{R}} e^{-is\Delta} F_s \, ds \right\rangle = \iint_{\mathbb{R}^2} \langle e^{-it\Delta} F_t, e^{-is\Delta} F_s \rangle \, dtds = \iint_{\mathbb{R}^2} \left\langle F_t, e^{i(t-s)\Delta} F_s \right\rangle \, dtds.$$

This expression effectively measures the interaction between all pairs (t, s), as highlighted earlier. Now if

$$G(x,t) = \int_{\mathbb{R}} e^{i(t-s)\Delta} F_s(x) \, ds,$$

we may write

$$\left\| \int e^{-it\Delta} F_t \right\|^2_{F^2_x} = \iint_{\mathbb{R}^2} \left\langle F_t, e^{i(t-s)\Delta} F_s \right\rangle \, dtds = \int_{\mathbb{R}} \left\langle F_t, \int_{\mathbb{R}} e^{i(t-s)\Delta} F_s \, ds \right\rangle \, dt = \int_{\mathbb{R}^d \times \mathbb{R}} F\bar{G}.$$

By Hölder,

$$\left\| \int e^{-it\Delta} F_t \right\|^2_{L^2_x} \leq \|F\|_{L^{\sigma'}_{x,t}} \left\|\bar{G}\right\|_{L^{\sigma}_{x,t}}.$$

Finally, by the Duhamel bound derived in the previous lecture, $\left\|\bar{G}\right\|_{L^{\sigma}_{x,t}} \lesssim \|F\|_{L^{\sigma'}_{x,t}}$. Hence

$$\left\| \int e^{-it\Delta} F_t \right\|^2_{L^2_x} \lesssim \|F\|^2_{L^{\sigma'}_{x,t}}.$$

□

In fact, Lemma 3 is closely related to the inhomogeneous Schrödinger equation, and is significant enough that it may be restated as its own theorem:

Theorem 4. *If $\partial_t u = i\Delta u + F$ with $F \in C^\infty_c(\mathbb{R}^d \times \mathbb{R})$ and u vanishes before the support of F,*

$$\|u(x,0)\|_{L^2_x} \lesssim \|F\|_{L^{\sigma'}_{x,t}}.$$

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