18.156 Lecture Notes

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Recall that last class we considered two different PDEs. The first was $\Delta u \pm |\nabla u|^2 = 0$. For this one, we have good global regularity and can solve the Dirichlet problem. The second PDE that we considered was $\Delta u \pm |\nabla u|^4 = 0$. Solutions to this PDE do not have good regularity, and we cannot solve the Dirichlet problem.

Let us think about the first PDE some more, and let us fix the sign as a plus:

$$\Delta u + |\nabla u|^2 = 0.$$

The key estimate that we will need to get good global regularity is the following:

Proposition 1 (Key Estimate). *If $u \in C^2(\overline{\Omega}), \Delta u + |\nabla u|^2 = 0, u = \varphi$ on $\partial\Omega$, then*

$$\|\partial_{nor} u\|_{C^0(\partial\Omega)} \leq g(\Omega, \|\varphi\|_{C^2(\partial\Omega)}).$$

We note that an immediate consequence of this proposition is that

$$\|\Delta u\|_{C^0(\partial\Omega)} \leq g(\Omega, \|\varphi\|_{C^2(\partial\Omega)}),$$

which is a step toward achieving some global regularity.

Last class, we started proving a comparison principle that will be useful in the proof of this proposition. Let us recall where we were and finish the proof:

Proposition 2 (Comparison principle). *If*

$$Qu = \sum_{i,j} a_{ij}(\nabla u)\partial_i\partial_j u + b(\nabla u),$$

where a_{ij} is positive definite, and a, b are C^1 functions of ∇u, then if $u, w \in C^2(\overline{\Omega}), u \leq w$ on $\partial\Omega$, $Qu \geq Qw$ on Ω, then $u \leq w$ on Ω.

Proof. Last week, we proved the strict version (if $Qu > Qw$). Now suppose that we only had that $Qu \geq Qw$. We can define a function

$$u_\epsilon := u + \epsilon e^{Mx_1} - \epsilon \sup_{z \in \Omega} |\epsilon e^{Mz_1}|.$$

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Then, $u_\epsilon \leq u \leq w$ on $\partial\Omega$. Furthermore, if we can show that for $\epsilon > 0$ sufficiently small, $Qu_\epsilon > Qu \geq Qw$ on Ω, by the strict case we would have that $u_\epsilon \leq w$ on Ω and by taking $\epsilon \to 0$, we would have $u \leq w$ and we would be done.

To show that $Qu_\epsilon > Qu$, we can look at $\frac{d}{d\epsilon}Qu_\epsilon$. We have that

$$\frac{d}{d\epsilon}Qu_\epsilon = a_{11}(\nabla u_\epsilon)\partial_1\partial_1(e^{Mx_1}) + \text{other terms.}$$

The other terms are bounded by $C(\Omega, u, a, b)M$ as they are linear combinations of terms $\partial_i e^{Mx_1}$. The first term we can bound below by $\tilde{C}(\Omega, u, a, b)M^2$. So for M large enough and ϵ small enough, we have that $\frac{d}{d\epsilon}Qu_\epsilon > 0$ and therefore $Qu_\epsilon > Qu \geq Qw$. $\qquad\square$

From the comparison principle, we have a couple of easy corollaries:

Corollary 3 (Uniqueness). *If $u, w \in C^2(\overline{\Omega})$, $Qu = Qw$ on Ω and $u = w$ on $\partial\Omega$, then $u = w$.*

Proof. From the comparison principle, we know that $u \leq w$ and $u \geq w$ on Ω, so $u = w$. $\qquad\square$

Corollary 4 (Maximum principle). *If $u \in C^2(\overline{\Omega})$, $Qu = 0$, then $\max_\Omega u \leq \max_{\partial\Omega} u$ (and a similar statement holds for the minimum.*

Proof. Let $w = \max_{\partial\Omega} u$. This is a constant function, so $Qw = 0 = Qu$, which implies that $\max_\Omega u \leq \max_\Omega w = \max_{\partial\Omega} u$. By setting $w = \min_{\partial\Omega} u$, we have the corresponding statement for the minimum. $\qquad\square$

Now, let us work on the proof of the key estimate. We will approach it via the barrier method. Suppose that $\Delta u + |\nabla u|^2 = 0$ on Ω, $u = \varphi$ on $\partial\Omega$.

Let N be a distance D neighborhood of $\partial\Omega$, for a $D > 0$ to be chosen later. Our aim is to construct a function $B : N \to \mathbb{R}$ such that

(i) $B = \varphi$ on $\partial\Omega$

(ii) $B \geq \|\varphi\|_{C^0}$ on the inner boundary

(iii) $\Delta B + |\nabla B|^2 \leq 0$ on N

If we were able to construct such a function, then since by the maximum principle $|u| \leq \|\varphi\|_0 \leq B$ on the inner boundary and therefore all of ∂N, an application of the comparison principle gives us that $n \leq B$ on N. Therefore, $\partial_{nor} u \leq \partial_{nor} B$ on $\partial\Omega$. If we then replace u with $-u$, we can repeat the same process to get a barrier $\tilde{B} \geq -u$, so $-\tilde{B} \leq u$, and $\partial_{nor} u \geq -\partial_{nor}\tilde{B}$. Since we

will construct B and $\tilde{B}$ so that their derivatives are bounded by $g(\Omega, \|\varphi\|_{C^2})$, this shows us that $\|\partial_{nor} u\|_{C^0(\partial\Omega)} \leq g(\Omega, \|\varphi\|_{C^2})$.

Let $d(x) = \text{dist}(x, \partial\Omega)$. From differential topology, we know that if D is sufficiently small, dependent on Ω, then $d \in C^\infty(N)$ and $|\nabla d| = 1$. Now, we can extend φ to a function $\Phi : N \to \mathbb{R}$ where $\|\Phi\|_{C^2} \leq C(\Omega)\|\varphi\|_{C^2}$ and $\|\Phi\|_{C^0} \leq \|\varphi\|_{C^0}$. We want to define B as

$$B(x) = \Phi(x) + \psi(d(x)),$$

for some function $\psi : [0, D] \to \mathbb{R}$. To ensure that B has the three properties listed above, we need to impose some restrictions on ψ:

(i) $\psi(0) = 0$

(ii) $\psi(D) \geq 2\|\varphi\|_{C^0}$

(iii) This one takes a bit of analysis/computation. We can see that

$$\nabla(\psi(d)) = \psi' \cdot \nabla d$$
$$\Delta(\psi(d)) = \text{div}(\psi' \cdot \nabla d) = \psi''|\nabla d|^2 + \psi'\nabla d.$$

Then, using that $|\nabla d| = 1$, we have that

$$\Delta B + |\nabla B|^2 \leq \Delta\Phi + \psi'' + \psi'\Delta d + 2|\nabla\Phi|^2 + 2|\psi'|^2 \leq \psi'' + A + A|\psi'|^2,$$

where A is a function of Ω and $\|\varphi\|_{C^2}$. So if we had that

$$\psi'' \leq -A - A|\psi'|^2,$$

then we would have property (iii), $\Delta B + |\nabla B|^2 \leq 0$ on N.

Let us now try to construct such a function ψ. We're going to construct a function ψ such that $\psi(0) = 0$, $\psi'(0) = s$, and $\psi'' = -A - A|\psi'|^2$. We can explicitly solve for such a function. Let $y = \psi'$. Then, we want that $y' = -A - Ay^2$, or that $\frac{y'}{1+y^2} = -A$. Solving, we have that

$$\arctan y(d) = -Ad + \arctan(s).$$

So, $y(d) = \tan(-Ad + \arctan(s))$. Now, we can let

$$\psi(d) = \int_0^d \tan(-Ax + \arctan(s)) \, dx.$$

Since for s large, we can make $\arctan(s)$ arbitrarily close to $\pi/2$, where $\tan$ is infinite, we have that for s large and D small, ψ and therefore B satisfies the three conditions.

Let us move on now to a brief discussion of the minimal surface equation, which our guest lecturer Spencer Hughes will be talking about in the next couple of lectures.

Suppose that we have a function $u : \Omega \to \mathbb{R}$. Then, assuming sufficient regularity, we can find the area of the graph of this function to be

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Let us see what happens when we perturb the function u a little bit.

Lemma 5. *If $u \in C^2(\overline{\Omega}), v \in C^2(\overline{\Omega})$, $v = 0$ on $\partial\Omega$, then*

$$\frac{d}{dt} A(u + tv)|_{t=0} = -\int_\Omega (Mu) \cdot v,$$

where M is some differential operator

A corollary of this lemma gives us a necessary condition for an area-minimizing function u with given boundary conditions.

Corollary 6. *If $A(u) \le A(w)$ for all $w \in C^2(\overline{\Omega})$, $w = u$ on $\partial\Omega$, then $Mu = 0$.*

Now, let us prove this lemma:

Proof. Fix a v and let $u_t := u + tv$. Then, we have that

$$\int_\Omega \frac{d}{dt}[(1 + |\nabla u_t|^2)^{1/2} = \int_\Omega \frac{1}{2}(1 + |\nabla u_t|^2)^{-1/2} \cdot 2\nabla u_t \cdot \nabla v.$$

At $t = 0$, this is equal to

$$\int_\Omega \sum_i \partial_i((1 + |\nabla u|^2)^{-1/2}\partial_i u) \cdot v.$$

$\square$

This is of the form $\partial_i(a_{ij}\partial_j u)$, where $a_{ij} = (1 + |\nabla u|^2)^{-1/2}\delta_{ij}$. We can also rewrite in a form so that we may use the comparison principle, $a_{ij}(\nabla u)\partial_i\partial_j u + b(\nabla u)$.

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