**MATH 18.152 COURSE NOTES - CLASS MEETING # 19**

**18.152 Introduction to PDEs**, Fall 2011 Professor: Jared Speck

Class Meeting # 19: Schrödinger's Equation

## 1. Introduction

*Schrödinger's equation* is the fundamental PDE of quantum mechanics. In the case of a single quantum particle, the unknown function is the *wave function* $\psi(t,x)$, which is a map from $\mathbb{R}^{1+n}$ into the complex numbers:

$$\psi : \mathbb{R}^{1+n} \to \mathbb{C}.$$

Above and throughout these notes, $t$ is the time coordinate, and $x = (x^1, \cdots, x^n)$ are the spatial coordinates. Schrödinger's equation is

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = V(t,x)\psi(t,x), \tag{1.0.1}$$

where $\Delta = \sum_{i=1}^n \partial_i^2$ is the usual Laplacian with respect to the spatial variables, and $V(t,x)$ is the *potential*, which models the interaction of the particle with its environment. In this course, we will mainly consider the case of *free particles*, in which $V = 0$ (i.e., the homogeneous Schrödinger equation). In the case of free particles, there is an important family of solutions to (1.0.1), namely the *free waves*. The free wave solutions provide some important intuition about how solutions to the homogeneous Schrödinger equation behave. To derive the free wave solutions, we first make the assumption that

$$\psi(t,x) = e^{i(\omega t - \xi\cdot x)}, \tag{1.0.2}$$

where $\cdot$ is the Euclidean dot product. Above, $\omega \in \mathbb{R}$ is the *frequency*, and $\xi \in \mathbb{R}^n$ is the *wave vector*. Note that (1.0.2) can be written as $e^{i|\xi|(\frac{\omega}{|\xi|}t - \frac{\xi}{|\xi|}\cdot x)}$, where $|\xi|$ is the Euclidean length of $\xi$. Since $\frac{\xi}{|\xi|}$ is a unit vector in $\mathbb{R}^n$, it therefore follows that the *speed* of the plane wave is

$$\frac{\omega}{|\xi|}. \tag{1.0.3}$$

Plugging (1.0.2) into (1.0.1), we derive the algebraic relation

$$-(\omega + \frac{|\xi|^2}{2})e^{i(\omega t + \xi\cdot x)} = 0, \tag{1.0.4}$$

which implies

$$\omega = -\frac{|\xi|^2}{2}, \tag{1.0.5}$$

$$\frac{\omega}{|\xi|} = -\frac{|\xi|}{2}. \tag{1.0.6}$$

These conditions are necessary and sufficient in order for the function given in (1.0.2) to solve (1.0.1) when $V = 0$. Note in particular that (1.0.6) shows that the speed of the plane wave solution *depends on* $|\xi|$, and in particular that larger $|\xi|'s$ lead to larger speeds. The dependence of the speed of the plane wave on $\xi$ is known as *dispersion*, and (1.0.5) is known as the *dispersion relation* of Schrödinger's equation.

Dispersion plays a very important role in the analysis of certain PDEs, and in particular Schrödinger's equation. Heuristically, one sometimes imagines that a "typical" solution to a dispersive PDE is composed of many free waves, each moving at a different speed and/or spatial direction (at least when the dispersion relation is non-trivial). The dispersive nature of the PDE suggests that the different free wave components in the solution should separate from each other. As we will see (see e.g. Theorem 2.1), this heuristic argument is sometimes rigorously borne out, and separation can cause the overall amplitude of the solution to decay in time (frequently at a rate of $t$ to some negative power).

## 2. The Fundamental Solution

We are now going to study the following global Cauchy problem for Schrödinger's equation:

$$i\partial_t\psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \tag{2.0.7a}$$

$$\psi(0,x) = \phi(x). \tag{2.0.7b}$$

Let's start by momentarily forgetting about the initial data and instead trying to find the fundamental solution $K(t,x)$ to equation (2.0.7a). We will precisely define the fundamental solution below; it is analogous to the fundamental solution for the heat equation. As we will see, the techniques from Fourier analysis that we have previously developed will allow us to derive the fundamental solution with relative ease. To this end, we set $\psi(t,x) = K(t,x)$, take the spatial Fourier of equation (2.0.7a), and use the Fourier transform property $(\partial_{\vec{\alpha}}K)^\wedge(t,\xi) = (2\pi i\xi)^{\vec{\alpha}}\hat{K}(t,\xi)$ (and in particular $(\Delta K)^\wedge(t,\xi) = -4\pi^2|\xi|^2\hat{K}(t,\xi)$) to deduce the following ODE for $\hat{K}(t,\xi)$ :

$$i\partial_t\hat{K}(t,\xi) - 2\pi^2|\xi|^2\hat{K}(t,\xi) = 0. \tag{2.0.8}$$

We rewrite (2.0.8) as

$$\partial_t \ln \hat{K}(t,\xi) = -2\pi^2 i|\xi|^2, \tag{2.0.9}$$

which can be easily integrated to give

$$\hat{K}(t,\xi) = Ce^{-2\pi^2 it|\xi|^2}, \tag{2.0.10}$$

where $C(\xi)$ is a constant that we have to calculate.

To calculate $C(\xi)$, we recall that we are ultimately trying to solve the following initial value problem for Schrödinger's equation:

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \tag{2.0.11a}$$

$$\psi(0,x) = \phi(x). \tag{2.0.11b}$$

Since $K(t,x)$ is supposed to be the fundamental solution, we would like (in analogy with the results of our study of the heat equation) the solution to (2.0.11a) - (2.0.11b) to be of the form

$$\psi(t,x) = (K(t,\cdot) * \phi(\cdot))(x). \tag{2.0.12}$$

Formally taking the Fourier transform of (2.0.12), using the fact that the Fourier transform turns convolutions into products, and using (2.0.10), we arrive at the formal relation

$$\hat{\psi}(t,\xi) = \hat{K}(t,\xi)\hat{\phi}(\xi) = C(\xi)e^{-2\pi^2 it|\xi|^2}\hat{\phi}(\xi). \tag{2.0.13}$$

Since (2.0.13) must in particular hold at $t = 0$, it is easy to see that

$$C(\xi) = 1. \tag{2.0.14}$$

Thus, the spatial Fourier transform of $K$ can be expressed as

$$\hat{K}(t,\xi) = e^{-2\pi^2 it|\xi|^2}. \tag{2.0.15}$$

In the next proposition, we make rigorous sense of the above formal calculations, and we calculate $K(t,x)$ from $\hat{K}(t,\xi)$.

**Proposition 2.0.1** (**Calculation of the Fundamental Solution** $K(t,x)$ **for Schrödinger's equation**). *Let $\phi(x)$ be a smooth compactly supported function, and let $\psi(t,x)$ be the function whose spatial Fourier transform is defined as in* (2.0.13)*:*

$$\hat{\psi}(t,\xi) = \hat{K}(t,\xi)\hat{\phi}(\xi), \tag{2.0.16}$$

*where $\hat{K}(t,\xi)$ is defined in* (2.0.15)*. Then if $t > 0$, we have that*

$$\psi(t,x) = (K(t,\cdot) * \phi)(x) \overset{def}{=} \int_{\mathbb{R}^n} K(t,y)\phi(x-y)\,d^n y = \int_{\mathbb{R}^n} K(t,x-y)\phi(y)\,d^n y, \tag{2.0.17}$$

*where*

$$K(t,x) = \frac{1}{(2\pi it)^{n/2}} e^{i\frac{|x|^2}{2t}}. \tag{2.0.18}$$

*Above, $i^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$.*

**Remark 2.0.1.** We refer to $\hat{K}(t,\xi)$ as the Fourier transform of $K(t,x)$, and $K(t,x)$ as the inverse Fourier transform of $\hat{K}(t,\xi)$.

**Remark 2.0.2.** Note that $K(t,\cdot)$ is not an element of $L^1$ because $\int_{\mathbb{R}} |K(t,x)|\, d^n x = \infty$. Since many of our previous results for the Fourier transform used the assumption that $K(t,\cdot) \in L^1$, our analysis of $K(t,x)$ is more delicate than these results.

*Proof.* For simplicity, let's consider only the case $n = 1$. Previously, we showed that since $\phi$ is smooth and compactly supported, $\hat{\phi}$ is smooth, is rapidly decaying at infinity, and is an element of $L^1$. Therefore, the same is true of the function $\hat{\psi}(\xi) = e^{-2\pi^2 it|\xi|^2}\hat{\phi}(\xi)$. Thus, by the Fourier inversion theorem, $\psi(t,x)$ is the inverse Fourier transform of $\hat{\psi}(t,\xi)$ :

$$\psi(t,x) = (\hat{\psi})^{\vee}(t,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{2\pi i \xi x}\hat{\psi}(t,\xi)\, d\xi = \int_{\mathbb{R}} e^{2\pi i \xi x} e^{-2\pi^2 it|\xi|^2}\hat{\phi}(\xi)\, d\xi. \tag{2.0.19}$$

To complete the proof, we will use that fact that the aforementioned properties of $\hat{\phi}$ together with the expression (2.0.19) allow us to express

$$\psi(t,x) = \lim_{\delta \to 0^+} \int_{\mathbb{R}} e^{2\pi i \xi x} e^{-2\pi^2(\delta + i)t|\xi|^2}\hat{\phi}(\xi)\, dx. \tag{2.0.20}$$

We will show (2.0.20) at the end of the proof; let us take it for granted at the moment.
Defining

$$f_{\delta;t}(\xi) \stackrel{\text{def}}{=} e^{-2\pi^2(\delta+i)t|\xi|^2}, \tag{2.0.21}$$

we see that (2.0.20) is by definition equivalent to

$$\psi(t,x) = \lim_{\delta \to 0^+} (f_{\delta;t}\hat{\phi})^{\vee}(x). \tag{2.0.22}$$

Note that $f_{\delta;t}$ is a Gaussian whose argument has *negative* real part. Thus, we have previously calculated its inverse Fourier transform:

$$f_{\delta;t}^{\vee}(x) = \frac{1}{\sqrt{2\pi(\delta+i)t}} e^{-|x|^2/(2t(\delta+i))}. \tag{2.0.23}$$

Furthermore, it is easy to see that

$$\lim_{\delta \to 0^+} f_{\delta;t}^{\vee}(x) = \frac{1}{\sqrt{2\pi i t}} e^{i|x|^2/(2t)}. \tag{2.0.24}$$

We note that in the formula (2.0.24), $\sqrt{i} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$.

Using (2.0.22), the Fourier transform + Fourier inversion identity $(uv)^{\vee} = [u^{\vee} * v^{\vee}]$, and the Fourier inversion theorem $(\hat{\phi})^{\vee} = \phi$, we have that

$$\begin{aligned} \psi(t,x) &= \lim_{\delta \to 0^+} [f_{\delta;t}^{\vee} * \phi](x) \stackrel{\text{def}}{=} \lim_{\delta \to 0^+} \int_{\mathbb{R}} f_{\delta;t}^{\vee}(x-y)\phi(y)\ dy \\ &= \int_{\mathbb{R}} \lim_{\delta \to 0^+} f_{\delta;t}^{\vee}(x-y)\phi(y)\ dy \\ &= \frac{1}{\sqrt{2\pi i t}} \int_{\mathbb{R}} e^{i|x-y|^2/(2t)}\phi(y)\ dy. \end{aligned} \tag{2.0.25}$$

We are allowed to bring the limit inside the integral in (2.0.25) because $\phi(y)$ is smooth and compactly supported and because (for each fixed $t > 0$) the limit (2.0.24) is achieved *uniformly* on compact spatial sets. We have thus shown (2.0.17).

It remains to prove (2.0.20). We need to show that

$$\Big| \int_{\mathbb{R}} e^{2\pi i \xi x} e^{-2\pi^2 it|\xi|^2} \big(e^{-2\pi^2 \delta t|\xi|^2} - 1\big)\hat{\phi}(\xi)\, d\xi \Big| \tag{2.0.26}$$

goes to 0 as $\delta \downarrow 0$. As we have previously discussed several times, the key to such an estimate is to split the integral over $\mathbb{R}$ into an integral over a ball $[-R, R]$ and its complement. More precisely, for any $R > 0$, the expression (2.0.26) can be bounded as follows:

$$\begin{aligned}
&\leq \int_{[-R,R]} |e^{-2\pi^2\delta t|\xi|^2} - 1||\hat{\phi}(\xi)|\, d\xi + \int_{\{|\xi| \geq R\}} \underbrace{|e^{-2\pi^2\delta t|\xi|^2} - 1|}_{\leq 1} |\hat{\phi}(\xi)|\, d\xi \\
&\leq \max_{\xi \in [-R,R]} |e^{-2\pi^2\delta t|\xi|^2} - 1| \int_{[-R,R]} |\hat{\phi}(\xi)|\, dx + \int_{\{|\xi| \geq R\}} |\hat{\phi}(\xi)|\, d\xi \\
&\stackrel{\text{def}}{=} I + II.
\end{aligned} \tag{2.0.27}$$

Let $\epsilon > 0$ be a positive number. In our previous studies of the Fourier transform, we showed that (see also the remarks above) $\int_{\mathbb{R}} |\hat{\phi}|\, d\xi \stackrel{\text{def}}{=} \|\hat{\phi}\|_{L^1} < \infty$. Now by Taylor expanding, we see that the following inequality holds whenever $R > 0$, $\xi \in [-R, R]$, and $\delta t R^2$ is sufficiently small:

$$|e^{-2\pi^2\delta t|\xi|^2} - 1| \leq C\delta t R^2, \tag{2.0.28}$$

where $C$ is a positive constant. Thus, we have the following estimate, valid whenever $\delta t R^2$ is sufficiently small:

$$|I| \leq C\delta t R^2 \int_{[-R,R]} |\hat{\phi}(\xi)|\, dx \leq C\epsilon t R^2 \|\hat{\phi}\|_{L^1}. \tag{2.0.29}$$

Furthermore, since $\|\hat{\phi}\|_{L^1} < \infty$, if $R$ is sufficiently large, then

$$|II| \leq \epsilon. \tag{2.0.30}$$

Thus, if $t$ is fixed, $R$ is first chosen to be sufficiently large, and then $\delta$ is chosen to be sufficiently small, we have that

$$|I| + |II| \leq C\delta t R^2 + \epsilon \leq 2\epsilon. \tag{2.0.31}$$

In total, we have shown that if $\delta$ is sufficiently small, then (2.0.26) is $\leq 2\epsilon$. Since this holds for any $\epsilon > 0$, we have thus shown (2.0.20).

□

We now formally define the fundamental solution.

**Definition 2.0.1 (The Fundamental Solution to Schrödinger's equation).** The fundamental solution associated to (1.0.1) is the function $K(t, x) = \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}}$ given in (2.0.18).

As an exercise, let's check that $K(t,x)$ verifies Schrödinger equation.

**Lemma 2.0.2** (**$K(t,x)$ verifies the free Schrödinger equation**)**.** *For $t > 0$, $K(t,x)$ is a solution to the free Schrödinger equation.*

*Proof.* We use the chain rule to calculate

$$\partial_j e^{i\frac{|x|^2}{2t}} = x^j \frac{i}{t} e^{i\frac{|x|^2}{2t}}, \tag{2.0.32}$$

$$\partial_j^2 e^{i\frac{|x|^2}{2t}} = \big(1 + \frac{i(x^j)^2}{t}\big)\frac{i}{t} e^{i\frac{|x|^2}{2t}}, \tag{2.0.33}$$

$$\frac{1}{2}\Delta K(t,x) = \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}} \big(i\frac{n}{2t} - \frac{|x|^2}{2t^2}\big) e^{i\frac{|x|^2}{2t}}, \tag{2.0.34}$$

$$i\partial_t K(t,x) = \frac{i}{(2\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}} \Big( -\frac{n}{2t} - \frac{i|x|^2}{2t^2}\Big). \tag{2.0.35}$$

From the last two calculations, it easily follows that

$$i\partial_t K(t,x) + \frac{1}{2}\Delta K(t,x) = 0. \tag{2.0.36}$$

□

We would like our fundamental solution to have the property that $\lim_{t\to 0^+}\psi(t,x) = \phi(x)$ for nice functions $\phi$, where $\psi(t,x) \overset{\text{def}}{=} [K(t,\cdot) * \phi(\cdot)](x)$. Now using (2.0.13), if the initial datum $\phi$ is smooth and compactly supported (and therefore, as previously shown, $\hat{\phi}$ is smooth and rapidly decaying), it is not difficult to show that

$$\lim_{t\downarrow 0} \|\hat{\psi}(t,\cdot) - \hat{\phi}\|_{L^2} = 0. \tag{2.0.37}$$

(2.0.13) shows that the transformed function $\hat{\psi}(t,\cdot)$ converges to the transformed datum $\hat{\phi}(\cdot)$ in the $L^2$ norm as $t \downarrow 0$. But how does the function $\psi(t,\cdot) \overset{\text{def}}{=} [K(t,\cdot) * \phi(\cdot)](x)$ behave as $t\downarrow 0$? By (2.0.17), this is equivalent to studying the behavior of $\frac{1}{(2\pi i t)^{n/2}}\int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{2t}}\phi(y)\,d^n y$ as $t\downarrow 0$. The next proposition briefly addresses this surprisingly difficult question.

**Proposition 2.0.3** (**The behavior of $K(t,\cdot) * \phi(\cdot)$ as $t\downarrow 0$**)**.** *Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\lim_{t\to 0^+} \frac{1}{(2\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{2t}} \phi(y)\, d^n y = \phi(x). \tag{2.0.38}$$

*Proof.* The proof of this proposition requires a technically involved technique from Fourier Analysis known as the method of stationary phase; it is therefore slightly beyond the scope of this course. The main difficulty is that the most of the important behavior in (2.0.38) is due to the rapid oscillation in $y$ of the integrand (except when $y$ is near $x$!) as $t\downarrow 0$. □

We are now ready to state and prove the main theorem concerning the solution to the free Schrödinger equation.

**Theorem 2.1 (The Solution to the Global Cauchy Problem Schrödinger's Equation and the Dispersive Estimate).** *Let $\phi(x) \in C_c^\infty(\mathbb{R}^n)$. Then there exists a unique solution $\psi \in C^\infty((0,\infty) \times \mathbb{R}^n)$ to the free Schrödinger equation*

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \qquad t > 0, x \in \mathbb{R}^n, \tag{2.0.39a}$$

$$\psi(0,x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{2.0.39b}$$

*The solution can be expressed as*

$$\psi(t,x) = [K(t,\cdot) * \phi(\cdot)](x), \tag{2.0.40}$$

*where $K(t,x)$ is the fundamental solution defined in* (2.0.18).

*Furthermore, for each $t > 0$, the solution $\psi(t,x)$ verifies the* ***dispersive estimate***

$$\|\psi(t,\cdot)\|_{C_0} \overset{def}{=} \max_{x \in \mathbb{R}^n} |\psi(t,x)| \leq \frac{C}{t^{n/2}} \|\phi\|_{L^1} \overset{def}{=} \frac{C}{t^{n/2}} \int |\phi(x)|\, d^n x. \tag{2.0.41}$$

*Above, $C > 0$ is a constant that does not depend on the initial data.*

*Proof.* Let $\mathcal{L} \overset{\text{def}}{=} i\partial_t + \frac{1}{2}\Delta_x$ denote the free Schrödinger operator. By definition, we have that

$$[K(t,\cdot) * \phi(\cdot)](x) = \int_{\mathbb{R}^n} \phi(y) \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x-y|^2}{2t}}\, d^n y. \tag{2.0.42}$$

According to our previously discussed differentiation-under-the-integral theorem (and making use of our assumptions on $\phi(x)$), for $t > 0$, we can differentiate under the integral in (2.0.42) and use Lemma 2.0.2 to deduce that

$$\mathcal{L}[K(t,\cdot) * \phi(\cdot)](x) = \int_{\mathbb{R}^n} \phi(y) \mathcal{L}\Big\{ \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x-y|^2}{2t}} \Big\}\, d^n y = 0. \tag{2.0.43}$$

Thus, $\phi * K_t$ verifies Schrödinger's equation (2.0.39a).

The fact that $\psi \in C^\infty((0,\infty) \times \mathbb{R}^n)$ follows from expressing

$$[K(t,\cdot) * \phi(\cdot)](x) = \int_{\mathbb{R}^n} \phi(x-y) \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|y|^2}{2t}}\, d^n y. \tag{2.0.44}$$

and repeatedly differentiating with respect to $x$ under the integral.

To prove (2.0.41), we note that the following simple pointwise inequality follows easily from (2.0.42):

$$\begin{aligned} |[K(t,\cdot) * \phi(\cdot)](x)| &\leq \left| \int_{\mathbb{R}^n} \phi(y) \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x-y|^2}{2t}}\, d^n y \right| \\ &\leq \frac{1}{(2\pi)^{n/2} t^{n/2}} \int_{\mathbb{R}^n} |\phi(y)|\, d^n y \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2} t^{n/2}} \|\phi\|_{L^1}. \end{aligned} \tag{2.0.45}$$

Taking the max over all $x \in \mathbb{R}^n$, the estimate (2.0.41) thus follows.

□

Let's now prove a very important property of sufficiently regular solutions to the free Schrödinger equation: their $L^2$ norm is constant in time.

**Proposition 2.0.4 (Preservation of $L^2$ norm).** *Under the assumptions of Theorem 2.1, we have that*

$$\|\psi(t,\cdot)\|_{L^2} = \overbrace{\|\phi\|_{L^2}}^{L^2 \text{ norm of the data}}, \tag{2.0.46}$$

*where the $L^2$ norm on the left-hand of* (2.0.46) *is taken over the spatial variables only. In particular, if $\int_{\mathbb{R}^n} |\phi(x)|^2 \, d^n x = 1$, then $\int_{\mathbb{R}^n} |\psi(t,x)|^2 \, d^n x = 1$ holds for all $t \geq 0$.*

*Proof.* We give two proofs, the first using the original solution, and the second using its Fourier transform; both proofs are important. For the first proof, we begin by noting that if

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \tag{2.0.47}$$

then by taking the complex conjugate of both sides, we have that

$$-i\partial_t \bar{\psi}(t,x) + \frac{1}{2}\Delta\bar{\psi}(t,x) = 0, \tag{2.0.48}$$

where $\bar{\psi}$ denotes the complex conjugate of $\psi$.

Differentiating under the integral in the definition of the $L^2$ norm, recalling that $|\psi|^2 = \psi\bar{\psi}$, and using (2.0.47) - (2.0.48), we thus deduce that

$$\begin{aligned} \frac{d}{dt}\|\psi(t,\cdot)\|_{L^2}^2 = \frac{d}{dt}\int_{\mathbb{R}^n} \psi(t,x)\bar{\psi}(t,x)\, d^n x &= \int_{\mathbb{R}^n} \partial_t\psi(t,x)\bar{\psi}(t,x) + \psi(t,x)\partial_t\bar{\psi}(t,x)\, d^n x \\ &= \frac{i}{2}\int_{\mathbb{R}^n} \Delta\psi(t,x)\bar{\psi}(t,x) - \psi(t,x)\Delta\bar{\psi}(t,x)\, d^n x. \end{aligned} \tag{2.0.49}$$

Integrating by parts on the right-hand side of (2.0.49), we conclude that

$$\frac{d}{dt}\|\psi(t,\cdot)\|_{L^2}^2 = -\frac{i}{2}\int_{\mathbb{R}^n} \nabla\psi(t,x)\cdot\nabla\bar{\psi}(t,x) - \nabla\psi(t,x)\cdot\nabla\bar{\psi}(t,x)\, d^n x = 0, \tag{2.0.50}$$

where $\cdot$ denotes the Euclidean dot product. We have thus shown (2.0.46).

For the second proof, we begin by recalling (2.0.13) and (2.0.14):

$$\hat{\psi}(t,\xi) = e^{-2\pi^2 it|\xi|^2}\hat{\phi}(\xi). \tag{2.0.51}$$

In particular, (2.0.51) implies that

$$|\hat{\psi}(t,\xi)|^2 = |\hat{\phi}(\xi)|^2. \tag{2.0.52}$$

Integrating (2.0.52) over $\mathbb{R}^n$, we deduce that

$$\|\hat{\psi}(t,\cdot)\|_{L^2} = \|\hat{\phi}\|_{L^2}, \tag{2.0.53}$$

where the $L^2$ norm on the left-hand side of (2.0.53) is taken over the $\xi$ variables only. Finally, by Plancherel's theorem, we see that (2.0.53) implies

$$\|\psi(t,\cdot)\|_{L^2} = \|\phi\|_{L^2}. \tag{2.0.54}$$

Again, we have shown (2.0.46).

$\square$

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