MATH 18.152 COURSE NOTES - CLASS MEETING # 2

18.152 Introduction to PDEs, Fall 2011 — Professor: Jared Speck

Class Meeting # 2: The Diffusion (aka Heat) Equation

1. Introduction to the Heat Equation

The heat equation for a function $u(t,x)$, $x \stackrel{\text{def}}{=} (x^1, \cdots, x^n) \in \mathbb{R}^n$, is

$$u_t - D\Delta u = f(t,x). \tag{1.0.1}$$

Here, the constant $D > 0$ is the *diffusion coefficient*, $f(t,x)$ is an inhomogeneous term, and Δ is the *Laplacian* operator, which takes the following form in Cartesian coordinates:

$$\Delta \stackrel{\text{def}}{=} \sum_{i=1}^{n} \partial_i^2. \tag{1.0.2}$$

Equation (1.0.1) is first-order and linear.

2. A simple model of heat flow that leads to the heat equation

We now give an example of a simple model of heat flow that leads to the heat equation. Consider a homogeneous, isotropic solid body $\mathcal{B} \subset \mathbb{R}^n$ ($n = 3$ is the physically relevant case) described by the following physical properties:

$$\rho \stackrel{\text{def}}{=} \text{mass density} \sim [\text{mass}] \times [\text{Volume}]^{-1} = \text{constant}, \tag{2.0.3}$$

$$e(t,x) \stackrel{\text{def}}{=} \text{thermal energy per unit mass} \sim [\text{energy}] \times [\text{mass}]^{-1}. \tag{2.0.4}$$

Let's also assume that heat is supplied to the body by an external source which pumps in heat at the following rate per unit mass:

$$\mathscr{R} \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{mass}]^{-1}. \tag{2.0.5}$$

The total thermal $E(t;V)$ energy contained in a body sub-volume $V \subset \mathcal{B}$ at time t is the integral of $e(t,x)$ over V:

$$E(t;V) \stackrel{\text{def}}{=} \int_V \rho e(t,x)\, d^n x. \tag{2.0.6}$$

The rate of change of the total energy contained in V is

$$\frac{d}{dt}E(t;V) = \frac{d}{dt}\int_V \rho e(t,x)\, d^n x = \int_V \rho \partial_t e(t,x)\, d^n x. \tag{2.0.7}$$

In (2.0.7), we have assumed that you can differentiate under the integral; we can do this when $e(t,x)$ is a "nice" function. We will be more precise about the meaning of "nice" later in the course.

Let's now address the factors that can cause $\frac{d}{dt}E(t;V)$ to be non-zero. That is, let's account for the factors that cause the energy within the volume V to change. In our simple model, we will account for only two factors. First, by integrating (2.0.5) over V, we deduce the rate of energy pumped into the sub-volume V by the external source:

$$\int_V \rho\mathscr{R}(t,x)\,d^n x \sim [\text{energy}] \times [\text{time}]^{-1}. \tag{2.0.8}$$

Second, we will also *assume* that heat energy is flowing throughout the body, and that flow can be modeled by a *heat flux vector* $\mathbf{q}$

$$\mathbf{q} \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{area}]^{-1}, \tag{2.0.9}$$

which specifies the direction and magnitude of heat flow across a unit area. That is, if $d\sigma \subset \partial V$ is a small surface area with *outward* unit-normal $\hat{\mathbf{N}}$, then $\mathbf{q}\cdot\hat{\mathbf{N}}$ is the energy flowing *out* of the small surface. Thus, the rate of heat energy flowing *into* V is

$$-\int_{\partial V} \mathbf{q}\cdot\hat{\mathbf{N}}\,d\sigma = -\int_V \nabla\cdot\mathbf{q}\,d^n x \sim [\text{energy}] \times [\text{time}]^{-1}, \tag{2.0.10}$$

where the equality follows from the divergence theorem.

We will connect the various energies together by *assuming* the following energy conservation "law:" *The rate of change of total energy in the sub-volume V is equal to the rate of heat energy flowing into V + rate of heat energy supplied by the external source.* Using (2.0.7), (2.0.8), and (2.0.10), we see that this "law" takes the following form in terms of integrals:

$$\int_V \rho\partial_t e(t,x)\,d^n x = -\int_V \nabla\cdot\mathbf{q}\,d^n x + \int_V \rho\mathscr{R}\,d^n x. \tag{2.0.11}$$

Since the above relations are assumed to hold for all body sub-volumes V, the integrands must be equal (again, as long as they are nice):

$$\rho\partial_t e(t,x) = -\nabla\cdot\mathbf{q} + \rho\mathscr{R}. \tag{2.0.12}$$

2.1. **Fourier's law.** In order to turn (2.0.12) into a PDE that we can study, we need to make another *assumption* about $e(t,x)$, $\mathbf{q}$, and their relation to the *temperature* $u(t,x)$. Fourier hypothesized the following "Fourier's Law of heat conduction:"

$$\mathbf{q}(t,x) = -\kappa\nabla u(t,x), \tag{2.1.1}$$

where $\kappa > 0$ is the *thermal conductivity*, and $\nabla u \stackrel{\text{def}}{=} (\partial_1 u, \cdots, \partial_n u)$ is the spatial derivative gradient of the temperature $u(t,x)$. We will assume that κ is a constant. Recall that at each fixed t, $\nabla u(t,x)$ points in the direction of *maximal increase* and that $\nabla u(t,x)$ is perpendicular to the level sets $\{x \mid u(x) = \text{constant}\}$. Thus, (2.1.1) states that heat flows "from hot to cold" (i.e. towards decreasing temperature) and that the flow is perpendicular to the surfaces of constant temperature.

Remark 2.1.1. (2.1.1) is *NOT A FUNDAMENTAL LAW OF NATURE*! It is a simple but reasonable (under certain circumstances) model!

We need one more assumption in order to derive our PDE - we need to relate $e(t,x)$ to $u(t,x)$. We will assume a very simple model, which is experimentally verified by many substances in moderate temperature ranges:

$$e = c_v u. \tag{2.1.2}$$

Here, $c_v > 0$ is the *specific heat at constant volume*. We also assume that c_v is constant. Like many of our previous assumptions, (2.1.2) is also just a simple model, and not a fundamental law of nature.

Finally, we combine (2.0.12), (2.1.1), and (2.1.2), and use the identity $\nabla \cdot \nabla u = \Delta u$, thus arriving at

$$\partial_t u(t,x) = \frac{\kappa}{c_v \rho}\Delta u + \frac{1}{c_v}\mathscr{R}. \tag{2.1.3}$$

This is the heat equation (1.0.1) with $D = \frac{\kappa}{c_v \rho}$ and $f = \frac{1}{c_v}\mathscr{R}$.

3. Well-posedness

Remember, one of the main goals of PDE theory is to figure out which kind of *data* lead to a unique solution. It is not always obvious which kind of data we are allowed to specify in order to solve the equation. When we have a PDE and a notion of data such that the data always lead to a unique solution, and the solution depends "continuously" on the data, we say that the *problem is well-posed*.

3.1. Dirichlet boundary conditions.

Let's study Dirichlet boundary conditions for the heat equation in $n = 1$ dimensions. Think of a one-dimensional rod with endpoints at $x = 0$ and $x = L$. Let's set most of the constants equal to 1 for simplicity, and assume that there is no external source pumping energy into the rod, i.e., that there is no inhomogeneous term f.

Then we could, for example, prescribe the temperature of the rod at $t = 0$ (sometimes called *Cauchy* data) and also at the boundaries $x = 0$ and $x = L$ *for all times* $t \in [0,T]$:

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(0,x) = g(x), & x \in [0,L], \qquad \text{(Cauchy data)}, \\ u(t,0) = h_0(t), & u(t,L) = h_L(t), \qquad t > 0, \qquad \text{(Dirichlet data)}. \end{cases} \tag{3.1.1}$$

As we will see, under suitable assumptions on the functions, g, h_0, h_L, these conditions lead to a well-posed problem.

3.2. Neumann (N for Normal!) boundary conditions.

Instead of prescribing the temperature at the boundaries, let's instead prescribe the *inward rate of heat flow* (given by Fourier's law with $\kappa = 1$) at the boundaries:

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(0,x) = g(x), & \text{(Cauchy data)}, \\ -\partial_x u(t,0) = h_0(t), & \partial_x u(t,L) = h_L(t), \qquad \text{(Neumann data)}. \end{cases} \tag{3.2.1}$$

Under suitable assumptions on the functions, g, h_0, h_L, these conditions also lead to a well-posed problem.

3.3. **Robin boundary conditions.** We can also take some linear combinations of the Dirichlet and Neumann conditions:

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(0,x) = g(x), & \text{(Cauchy data)}, \\ -\partial_x u(t,0) + \alpha u(t,0) = h_0(t), \qquad \partial_x u(t,L) + \alpha u(t,L) = h_L(t), & \text{(Robin data)}, \end{cases} \tag{3.3.1}$$

where $\alpha > 0$ is a *positive* constant. Under suitable assumptions on the functions, g, h_0, h_L, these conditions also lead to a well-posed problem.

3.4. **Mixed boundary conditions.** The above three boundary conditions are called *homogeneous* because they are of the same type at each end. It is also possible to prescribe one condition at one endpoint, and a different condition at the other endpoint. These are called *mixed boundary conditions*. These conditions also lead to a well-posed problem.

4. Separation of variables

We now discuss a technique, known as *separation of variables*, that can be used to explicitly solve certain PDEs. It is especially useful in the study of linear PDEs. Although this technique is applicable to some important PDEs, it is unfortunately far from universally applicable.

In a nutshell, the separation of variables technique can be summarized as:

- Look for a solution of the form $u(t,x) = v(t)w(x)$.
- Plug this guess into the PDE and hope that the PDE forces the functions v and w to be solutions to ODEs that can be solved without too much trouble.

As we will see, when one tries to apply this technique, one quickly runs into difficulties that are best addressed using techniques from Fourier analysis. We don't have time right now to give a detailed introduction to Fourier analysis, but we will return to it later in the course if time permits; at the moment, we will only show how to use some of these techniques, without fully justifying them.

A great way to illustrate separation of variables is through an example. Let's try to solve the heat equation problem with homogeneous (i.e., vanishing) Dirichlet conditions

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in (0,T] \times [0,1], \\ u(0,x) = x, & x \in [0,1], \\ u(t,0) = 0, & u(t,1) = 0, \end{cases} \tag{4.0.1}$$

by separation of variables.

Remark 4.0.1. Note that such a solution cannot possibly be continuous at the point $(0,1)$.

We plug in the form $u(t,x) = v(t)w(x)$ into (4.0.1) and discover that

$$\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}. \tag{4.0.2}$$

This should hold for all t, x. It therefore must be the case that both sides are equal to a constant, which we will call λ. We then have

$$v'(t) = \lambda v(t), \tag{4.0.3a}$$

$$w''(x) = \lambda w(x). \tag{4.0.3b}$$

Furthermore, $w(0) = w(1) = 0$ by the boundary conditions.

Let's address v first, since it requires less work to deal with than w. If $\lambda \in \mathbb{R}$, then (4.0.3a) can be generally solved:

$$v(t) = Ae^{\lambda t} \tag{4.0.4}$$

for some $A \in \mathbb{R}$.

In contrast, the study of $w(x)$ splits into three cases:

- $\lambda = 0$. Then $w(x) = Bx + C$ for some $B, C \in \mathbb{R}$. The boundary conditions imply that $C = 0$ and $B + C = 0$, so that $B = C = 0$. Thus, this solution is not very interesting.
- $\lambda > 0$. Then $w(x) = Be^{\sqrt{\lambda}x} + Ce^{-\sqrt{\lambda}x}$ for some $B, C \in \mathbb{R}$. The boundary conditions imply that $B + C = 0$, and $Be^{\sqrt{\lambda}} + Ce^{-\sqrt{\lambda}} = 0$, which forces $B = C = 0$. This solution is also not very interesting.
- $\lambda < 0$. Then $w(x) = B\sin(\sqrt{|\lambda|}x) + C\cos(\sqrt{|\lambda|}x)$ for some $B, C \in \mathbb{R}$. The boundary condition $w(0) = 0$ forces $C = 0$, so $w(x) = B\sin(\sqrt{\lambda}x)$. The boundary condition $w(1) = 0$ then forces $\lambda = -\pi^2 m^2$ for some $m \in \mathbb{Z}^+$, where $\mathbb{Z}^+ \overset{\text{def}}{=}$ the set of *non-negative* integers. The λ are called *eigenvalues*, and the corresponding w_m are the corresponding *eigenvectors*. Equation (4.0.3a) is called an *eigenvalue problem* corresponding to the linear operator $\mathcal{L} \overset{\text{def}}{=} \partial_x^2$.

We have shown that the only solutions w are of the form $w_m(x) = B\sin(2\pi mx)$, $m \in \mathbb{Z}^+$. Using also (4.0.4) and the fact that $\lambda = -\pi^2 m^2$ for our solutions, we have produced a family of solutions to the heat equation $\partial_t u - \partial_x^2 u = 0$ that satisfying the boundary conditions:

$$u_m(t, x) = e^{-m^2\pi^2 t}\sin(m\pi x), \qquad A_m \in \mathbb{R}, \qquad m \in \mathbb{Z}^+. \tag{4.0.5}$$

But we haven't yet satisfied the initial condition $u(0, x) = x$. To do this, we could try using the superposition principle:

$$u(t, x) = \sum_{m=1}^{\infty} A_m u_m(t, x). \tag{4.0.6}$$

We would have to solve for the A_m to achieve the desired initial condition $u(0, x) = x$.

Here is a list of things we would have to do to fully solve this problem using this technique:

(1) Find plausible A_m.
(2) Show that the infinite sum (4.0.6) converges.
(3) Show that the infinite sum solves the heat equation.
(4) Show that $u(t, x)$ satisfies the boundary conditions.
(5) Check that $\lim_{t\to 0^+} u(t, x) = u(0, x) = x$. We also have to investigate in which sense this limit may or may not hold. We already know that this equality cannot hold pointwise at the point $(0, 1)$.

(6) Show that there can be no other solution with these initial/boundary conditions (uniqueness).

Let's deal with (1) first. If (4.0.6) holds, then at $t = 0$:

$$x = u(0, x) = \sum_{m=1}^{\infty} A_m u_m(0, x) = \sum_{m=1}^{\infty} A_m \sin(m\pi x). \tag{4.0.7}$$

This is a *Fourier series* expansion for the function $f(x) = x$ on the interval $[0, 1]$.

It is helpful to think of a function $f(x)$ as a vector in an infinite dimensional vector space and the $\sin(m\pi x)$ as basis vectors (however, it is not trivial to show that they form a basis...). Furthermore, if we introduce the dot product

$$\langle f(x), g(x)\rangle \overset{\text{def}}{=} \int_{[0,1]} f(x)g(x)\,dx, \tag{4.0.8}$$

then the basis vectors are orthogonal (do the computation yourself!):

$$\langle \sin(m\pi x), \sin(\pi n x)\rangle = \begin{cases} 1/2 \text{ if } m = n \\ 0 \text{ if } m \neq n. \end{cases} \tag{4.0.9}$$

This *suggests* that the following heuristic computations might be able to be made completely rigorous:

$$\begin{aligned} \int_{[0,1]} f(x)\sin(\pi n x)\,dx = \langle f(x), \sin(\pi n x)\rangle &= \langle \sum_{m=1}^{\infty} A_m \sin(m\pi x), \sin(\pi n x)\rangle \\ &= \sum_{m=1}^{\infty} \langle A_m \sin(m\pi x), \sin(\pi n x)\rangle \\ &= \frac{1}{2}A_n. \end{aligned} \tag{4.0.10}$$

Applying this to our function $f(x) = x$, we integrate by parts to compute that

$$A_m = 2\int_{[0,1]} x\sin(m\pi x)\,dx = -\frac{2}{m\pi}x\cos(m\pi x)|_{x=0}^{x=1} + \frac{2}{m\pi}\int_{[0,1]}\cos(\pi n x)\,dx = (-1)^{m+1}\frac{2}{m\pi}. \tag{4.0.11}$$

We now hope that our solution is:

$$u(t, x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x). \tag{4.0.12}$$

Remark 4.0.2. The individual terms $(-1)^{m+1}e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$ are sometimes called the *modes* of the solution. Note that each mode is rapidly decaying at an exponential rate as $t \to \infty$. Furthermore, the infinite sum $\sum_{m=1}^{\infty}(-1)^{m+1}e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$ also decays exponentially in time. Later in the course we will study the heat equation on all of $\mathbb{R}$, and we will once again see that under suitable assumptions, solutions to the heat equation tend to exponentially decay in time. However, if we had non-zero Dirichlet conditions for the problem (4.0.1), then the solution might not decay to 0, but instead to some other state.

Let's now answer some of the remaining questions from above.

(2) Thanks to the rapidly decaying in m factor $e^{-m^2\pi^2 t}$, for any $t > 0$, the series (4.0.12) can be seen to uniformly converge for $x \in [0,1]$ using one of the standard convergence arguments from analysis (carefully work through this argument yourself; pg. 9 of your book might be a helpful reference). The argument for $t = 0$ is much more subtle and is addressed in Theorem 4.1 below.

(3) We already know that each mode in (4.0.12) solves the heat equation. So what about the infinite sum? Again, for any $t > 0$, the $e^{-m^2\pi^2 t}$ factor plus standard results from analysis allow us to repeatedly differentiate the series term-by-term in both t and x (work through this yourself). In particular, the series is *smooth* (i.e., infinitely differentiable in all variables) for any $t > 0$. In particular, for $t > 0$, we have that

$$\partial_t u = \sum_{m=1}^{\infty} \partial_t[(-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)] = \sum_{m=1}^{\infty} (-1)^m m\pi e^{-m^2\pi^2 t} \sin(m\pi x), \tag{4.0.13a}$$

$$\partial_x^2 u = \sum_{m=1}^{\infty} \partial_x^2[(-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)] = \sum_{m=1}^{\infty} (-1)^m m\pi e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x), \tag{4.0.13b}$$

which shows that $-\partial_t u + \partial_x^2 u = 0$.

(4) The fact that u verifies the correct Dirichlet conditions at $x = 0$ and $x = 1$ follows from the fact that each of the modes does.

The remaining two questions require more work. We first quote the following theorem from Fourier analysis to help us understand the Fourier expansion at $t = 0$. Using this theorem, you will address question (5) in your homework.

Theorem 4.1 (**Some basic facts from Fourier analysis**). *If $f(x)$ is a function such that $\|f\|_{L^2([0,1])}^2 \overset{def}{=} \int_0^1 |f(x)|^2 dx < \infty$, then $f(x)$ can be Fourier-expanded as $f(x) = \sum_{m=1}^{\infty} A_m \sin(m\pi x)$, where $A_m = 2\int_{[0,1]} f(x)\sin(m\pi x)\, dx$. The infinite sum converges in the sense that*

$$\|f - \sum_{m=1}^{N} A_m \sin(m\pi x)\|_{L^2([0,1])} \to 0 \text{ as } N \to \infty. \tag{4.0.14}$$

We also have the Parseval identity

$$\|f\|_{L^2([0,1])}^2 = \sum_{m=1}^{\infty} A_m^2 \|\sin(m\pi x)\|_{L^2([0,1])}^2 = \sum_{m=1}^{\infty} \frac{1}{2} A_m^2. \tag{4.0.15}$$

Note that (4.0.15) *is an "infinite dimensional Pythagorean theorem."*

Furthermore, if f is continuous on $[0,1]$, then for any subinterval $[a,b] \subset (0,1)$,

$$\|f - \sum_{m=1}^{N} A_m \sin(m\pi x)\|_{C^0([a,b])} \to 0 \text{ as } N \to \infty, \tag{4.0.16}$$

i.e., the convergence is uniform on any closed subinterval $[a,b]$ of the open interval $(0,1)$.

Exercise 4.0.1. Many extensions of Theorem 4.1 are possible. Read Appendix A of your textbook in order to learn about them.

18.152 Introduction to Partial Differential Equations.
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