MATH 18.152 - FINAL EXAM

18.152 Introduction to PDEs, Fall 2011 — Professor: Jared Speck

Final Exam, **Monday, December 19**

Name:

Problem Number	Points	Score
I	20	
II	20	
III	30	
IV	20	
V	20	
VI	30	
VII	30	
Total	170	

Answer questions I - VII below. The point values are listed in the table above. Partial credit may be awarded, but only if you show all of your work and it is in a logical order. In order to receive credit, whenever you make use of a theorem/ proposition, make sure that you state it by name. Also, clearly state the hypotheses that are needed to the apply theorem/ proposition, and explain why the hypotheses are satisfied. You are allowed to use one handwritten page of notes (the front and back of an 8.5 × 11 inch sheet of white printer paper). No other books, notes, or calculators are allowed.

I. (20 points) Let $R > 0$ be a real number, and let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions that vanish whenever $|x| \geq R$. Let $\phi(t, x)$ be the solution to the following global Cauchy problem:

$$-\partial_t^2 \phi + \partial_x^2 \phi = 0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}, \tag{1}$$

$$\phi(0, x) = f(x), \qquad x \in \mathbb{R}, \tag{2}$$

$$\partial_t \phi(0, x) = g(x), \qquad x \in \mathbb{R}. \tag{3}$$

Show that $\phi(t, x) = 0$ whenever $|x| \geq R + t$ (for positive t only).

II. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) \stackrel{\text{def}}{=} \begin{cases} 1, & |x| \leq 2, \\ 0, & |x| > 2. \end{cases} \tag{1}$$

a) (10 points) Show that

$$\hat{f}(\xi) = 4\text{sinc}(4\xi), \tag{2}$$

where sinc : $\mathbb{R} \to \mathbb{R}$ is the function defined by

$$\text{sinc}(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{3}$$

b) (10 points) Compute $\|\hat{f}\|_{L^2}$.

III. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth compactly supported function. Let $u(t,x)$ be the unique smooth solution to the following global Cauchy problem:

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{1}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}^n, \tag{2}$$

$$\partial_t u(0,x) = 0, \qquad x \in \mathbb{R}^n, \tag{3}$$

where $\Delta \overset{\text{def}}{=} \sum_{j=1}^n \partial_j^2$ is the standard Laplacian with respect to the spatial coordinates $(x^1, \cdots, x^n)$. Let

$$\hat{u}(t,\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x)\, d^n x \tag{4}$$

be the Fourier transform of $u(t,x)$ with respect to the spatial variables only.
a) (5 points) Show that $\hat{u}(t,\xi)$ is a solution to the following initial value problem:

$$\partial_t^2 \hat{u}(t,\xi) = -4\pi^2 |\xi|^2 \hat{u}(t,\xi), \qquad (t,\xi) \in [0,\infty) \times \mathbb{R}^n, \tag{5}$$

$$\hat{u}(0,\xi) = \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{6}$$

$$\partial_t \hat{u}(0,\xi) = 0, \qquad \xi \in \mathbb{R}^n. \tag{7}$$

b) (10 points) Explicitly solve the above initial value problem. That is, find an expression for the solution $\hat{u}(t,\xi)$ in terms of $\hat{f}(\xi)$ (and some other functions of (t,ξ)).
Hint: If done correctly *and simplified*, your answer should involve a trigonometric function.

c) (5 points) Using part **b)** and the properties of the Fourier transform, express both $\partial_t \hat{u}(t,\xi)$ and $(\nabla u)^\wedge(t,\xi)$ in terms of $\hat{f}(\xi)$ (and some other functions of (t,ξ)). Here, $\nabla u(t,x) = (\partial_1 u(t,x), \partial_2 u(t,x), \cdots, \partial_n u(t,x))$ is the *spatial* gradient of $u(t,x)$.

d) (10 points) Using part **c)** and Fourier transform techniques (no integration by parts), show that for all $t \geq 0$, we have

$$\||Du(t,\cdot)|\|_{L^2} = \||\nabla f|\|_{L^2}, \tag{8}$$

where $Du \overset{\text{def}}{=} (\partial_t u, \partial_1 u, \partial_2 u, \cdots, \partial_n u)$ is the spacetime gradient of u, $|Du| \overset{\text{def}}{=} \sqrt{(\partial_t u)^2 + \sum_{j=1}^n (\partial_j u)^2}$ is the Euclidean norm of Du, $\nabla f = (\partial_1 f, \partial_2 f, \cdots, \partial_n f)$ is the spatial gradient of f, $|\nabla f| \overset{\text{def}}{=} \sqrt{\sum_{j=1}^n (\partial_j f)^2}$ is the Euclidean norm of ∇f, and the L^2 norm on the left-hand side of (8) is taken with respect to the spatial variables only.

IV. (20 points) Let $f : [0,1] \to \mathbb{R}$ be a smooth function. Let $u(t,x)$ be the unique smooth solution to the following inhomogeneous global Cauchy problem:

$$\partial_t u(t,x) - \partial_x^2 u(t,x) = -(t^2 + x^2), \qquad (t,x) \in [0,2] \times [0,1], \tag{1}$$

$$u(0,x) = f(x), \qquad x \in [0,1]. \tag{2}$$

Define

$$M \overset{\text{def}}{=} \max_{(t,x) \in [0,2] \times [0,1]} u(t,x). \tag{3}$$

Let $(t_0, x_0) \in (0,2) \times (0,1)$ $\big($i.e., (t_0, x_0) belongs to the interior of $[0,2] \times [0,1]\big)$.

Show that $u(t_0, x_0) = M$ is **impossible**.

V. Let (t, x) denote standard coordinates on $\mathbb{R}^{1+n}$, where t denotes the time coordinate and $x = (x^1, \cdots, x^n)$ denotes the spatial coordinates. Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be a smooth, compactly supported function. Let $\psi : \mathbb{R}^{1+n} \to \mathbb{C}$ be a solution to the following global Cauchy problem:

$$i\partial_t \psi(t, x) + \frac{1}{2}\Delta\psi(t, x) = 0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}^n, \tag{1}$$

$$\psi(0, x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{2}$$

a) (10 points) Show that

$$\|\psi(t, \cdot)\|_{L^2} = \|\phi\|_{L^2} \overset{\text{def}}{=} \sqrt{\int_{\mathbb{R}^n} |\phi(x)|^2 \, d^n x} \tag{3}$$

holds for all $t \geq 0$. On the left-hand side of (3), the L^2 norm of ψ is taken with respect to the spatial variables only.

b) (10 points) Use part **a)** to show that solutions to (1) - (2) are unique (i.e., that there is at most one smooth solution to the initial value problem (1) - (2)).

VI. Let $m_{\mu\nu} = \text{diag}(-1, 1, 1, \cdots, 1)$ denote the standard Minkowski metric. Let $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ be a field. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(m^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{4}\phi^4. \tag{1}$$

a) (5 points) Write down the Euler-Lagrange equation corresponding to (1).
b) (15 points) Compute the energy-momentum $T^{\mu\nu}$ corresponding to (1) and show that $T^{00} \geq 0$.
c) (5 points) Assume that ϕ is a C^2 solution to the Euler-Lagrange equation. Calculate $\partial_\mu T^{\mu\nu}$.
d) (5 points) Explain how the vectorfield $J^\mu \stackrel{\text{def}}{=} T^{\mu 0}$ can be used to derive a "useful" conserved (in time) quantity for C^2 solutions to the Euler-Lagrange equation.

VII. Respond to the following 6 short-answer questions.

a) (5 points) Give an example of a dispersive PDE.

b) (5 points) Give an example of an initial value problem PDE whose solutions do not propagate at finite speeds.

c) (5 points) Let $\Omega \subset \mathbb{R}^3$ be a domain, and let Δ denote the standard Laplacian on $\mathbb{R}^3$. The Green's function $G : \Omega \times \Omega \to \mathbb{R}$ is a function $G(x, y)$ that satisfies an inhomogeneous PDE with certain boundary conditions. Write down that PDE and also the boundary conditions.

d) (5 points) Classify the following PDE as elliptic, hyperbolic, or parabolic:

(1) $$-\partial_t^2 u(t,x) + 4\partial_t \partial_x u(t,x) - \partial_x^2 u(t,x) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

e) (5 points) Explain what it means for a PDE problem to be *well-posed*.

f) (5 points) Give an example of a liner PDE on $\mathbb{R}^2$ whose corresponding Cauchy problem (i.e., the initial value problem) is **not** well-posed.

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