# Lecture 23: Moser's approach to the mean value inequality

## 1 The mean value inequality: Moser's Approach

In this lecture we will give an alternative proof of the mean value inequality. As before we take $L$ a uniformly elliptic second order operator in divergence form, and $u$ an $L$ harmonic function on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$, take an open ball $B_s(x_0)$,$k > 1$ a positive constant, and $\eta$ be a cutoff function (ie $\eta : B_s(x_0) \to \mathbb{R}$ with $\eta = 0$ on the boundary). We have

$$\int_{B_s(x_0)} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta^2 u^k}{\partial x_j} = 0 \tag{1}$$

(this is from the weak definition of $L$ harmonic). Therefore

$$0 = k \int_{B_s(x_0)} \eta^2 u^{k-1} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2 \int_{B_s(x_0)} \eta u^k A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j}. \tag{2}$$

Note that

$$A_{ij} \frac{\partial u^\beta}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} = \beta^2 u^{2\beta-2} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \tag{3}$$

for all constants $\beta$. If we pick $\beta = \frac{k+1}{2}$we get

$$\left( \frac{k+1}{2} \right)^2 u^{k-1} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = A_{ij} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_i} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_j} \tag{4}$$

and applying this to 2 gives

$$\frac{4k}{(k+1)^2} \int_{B_s(x_0)} \eta^2 A_{ij} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_i} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_j} = -2 \int_{B_s(x_0)} \eta u^k A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j}. \tag{5}$$

Let $k = l_1 + l_2$ and apply Cauchy Schwarz to the right hand side to get

$$\frac{4k}{(k+1)^2} \int_{B_s(x_0)} \eta^2 A_{ij} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_i} \frac{\partial u^{\frac{k+1}{2}}}{\partial x_j} \leq 2 \left( \int_{B_s(x_0)} A_{ij} \eta^2 u^{2l_1} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right)^{1/2} \left( \int_{B_s(x_0)} A_{ij} u^{2l_2} \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right)^{1/2}. \tag{6}$$

Apply 3 with $\beta = l_1 + 1$ to get

$$\frac{4k}{(k+1)^2}\int_{B_s(x_0)} \eta^2 A_{ij}\frac{\partial u^{\frac{k+1}{2}}}{\partial x_i}\frac{\partial u^{\frac{k+1}{2}}}{\partial x_j} \le \frac{2}{l_1+1}\left(\int_{B_s(x_0)} A_{ij}\eta^2\frac{\partial u^{l_1+1}}{\partial x_i}\frac{\partial u^{l_1+1}}{\partial x_j}\right)^{1/2}\left(\int_{B_s(x_0)} A_{ij}u^{2l_2}\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}\right)^{1/2}. \tag{7}$$

Choosing $l_1 = \frac{k-1}{2}, l_2 = \frac{k+1}{2}$ we have

$$\frac{2l_2-1}{{l_2}^2}\int_{B_s(x_0)} \eta^2 A_{ij}\frac{\partial u^{l_2}}{\partial x_i}\frac{\partial u^{l_2}}{\partial x_j} \le \frac{2}{l_2}\left(\int_{B_s(x_0)} A_{ij}\eta^2\frac{\partial u^{l_2}}{\partial x_i}\frac{\partial u^{l_2}}{\partial x_j}\right)^{1/2}\left(\int_{B_s(x_0)} A_{ij}u^{2l_2}\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}\right)^{1/2}, \tag{8}$$

so dividing and squaring gives

$$\left(\frac{2l_2-1}{l_2}\right)^2\int_{B_s(x_0)} \eta^2 A_{ij}\frac{\partial u^{l_2}}{\partial x_i}\frac{\partial u^{l_2}}{\partial x_j} \le 4\int_{B_s(x_0)} A_{ij}u^{2l_2}\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}. \tag{9}$$

Use uniform ellipticity to simplify this to

$$\left(\frac{2l_2-1}{l_2}\right)^2\int_{B_s(x_0)} \eta^2|\nabla u^{l_2}|^2 \le \frac{4\Lambda}{\lambda}\int_{B_s(x_0)} u^{2l_2}|\nabla\eta|^2. \tag{10}$$

Since $k \ge 1$, we have $l_2 \ge 1$ and so $\left(\frac{2l_2-1}{l_2}\right)^2 \ge 1$. Thus

$$\int_{B_s(x_0)} \eta^2|\nabla u^{l_2}|^2 \le \frac{4\Lambda}{\lambda}\int_{B_s(x_0)} u^{2l_2}|\nabla\eta|^2. \tag{11}$$

We need to estimate $\int_{B_s(x_0)}|\nabla(\eta u^{l_2})|^2$. Note that

$$|\nabla(\eta u^{l_2})|^2 \le 2u^{2l_2}|\nabla\eta|^2 + 2\eta^2|\nabla u^{l_2}|^2. \tag{12}$$

Apply 11 to get

$$\int_{B_s(x_0)}|\nabla(\eta u^{l_2})| \le \left(2+\frac{8\Lambda}{\lambda}\right)\int_{B_s(x_0)} u^{2l_2}|\nabla\eta|^2. \tag{13}$$

From this we can use Sobolev to estimate

$$\begin{aligned}\left(\int_{B_s(x_0)}(\eta u^{l_2})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\le c\int_{B_s(x_0)}|\nabla(\eta u^{l_2})|^2 &(14)\\ &\le c\left(2+\frac{8\Lambda}{\lambda}\right)\int_{B_s(x_0)} u^{2l_2}|\nabla\eta|^2 &(15)\end{aligned}$$

for some dimensional constant $c$. Pick $r < s$ and let $\eta$ be the usual linear cutoff function, i.e.

$$\phi = \begin{cases} 1 & \text{on } B_r(x_0) \\ \frac{s-|x|}{s-r} & \text{on } B_s(x_0) \setminus B_r(x_0), \text{ and} \\ 0 & \text{outside } B_s(x_0) \end{cases} . \tag{16}$$

This gives

$$\begin{aligned} \left( \int_{B_r(x_0)} (u^{l_2})^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} &\leq \left( \int_{B_s(x_0)} (\eta u^{l_2})^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} & (17) \\ &\leq \tilde{c} \int_{B_s(x_0) \setminus B_r(x_0)} u^{2l_2} |\nabla \eta|^2 & (18) \\ &\leq \frac{\tilde{c}}{(s-r)^2} \int_{B_s(x_0) \setminus B_r(x_0)} u^{2l_2} & (19) \\ &\leq \frac{\tilde{c}}{(s-r)^2} \int_{B_s(x_0)} u^{2l_2} & (20) \end{aligned}$$

for an appropriate constant $\tilde{c}$ that depends on $L$ and $n$. Define $\chi = \frac{n}{n-2} > 1$ and $p = 2l_2$. We've shown that

$$\left( \int_{B_r(x_0)} u^{p\chi} \right)^{1/p\chi} \leq \left( \frac{\tilde{c}}{(s-r)^2} \right)^{1/p} \left( \int_{B_s(x_0)} u^p \right)^{1/p} . \tag{21}$$

We can define the $L^q$ norm of a function $f$ by

$$||f||_{L^q(B_R(x_0))} = \left( \int_{B_R(x_0)} f^q \right)^{1/q} . \tag{22}$$

This is useful notion of length for integrable functions, and in this notation we have

$$||u||_{L^{p\chi}(B_r(x_0))} \leq \left( \frac{\tilde{c}}{(s-r)^2} \right)^{1/p} ||u||_{L^p(B_s(x_0))} \tag{23}$$

for all $r < s$. Let $r_m = 1 + 2^{-m}$ and $p_m = 2\chi^m$. Then

$$||u||_{L^{p_{m+1}}(B_{r_{m+1}}(x_0))} \leq \left( \tilde{c} 2^{2m+2} \right)^{1/p_m} ||u||_{L^{p_m}(B_{r_m}(x_0))}, \tag{24}$$

so, by induction,

$$\begin{aligned} ||u||_{L^{p_{m+1}}(B_{r_{m+1}}(x_0))} &\leq \left( \Pi_{i=0}^m (2^{2i+2} \tilde{c})^{\frac{1}{2\chi^i}} \right) ||u||_{L^2(B_2(x_0))} & (25) \\ &\leq 2^{\sum_{i=0}^m \frac{i+1}{2\chi^m}} c^{\sum_{i=0}^m \frac{c}{2\chi^m}} ||u||_{L^2(B_2(x_0))}. & (26) \end{aligned}$$

Both of these sums converge as $m \to \infty$ by the ration test. Therefore if the $L_2$ norm of $u$ is finite on $B_2(x_0)$ then the $L_\infty$ norm on$B_1(x_0)$ is finite, and indeed, is bounded by a dimensional constant.