18.112 Functions of a Complex Variable
Fall 2008

Lecture 3: Analytic Functions; Rational Functions

(Text 21-32)

Remarks on Lecture 3

▶ Formula (14) on p.32 was proved under the assumption that $R(\infty) = \infty$. On the other hand, if $R(\infty)$ is finite, then (12) holds with $G \equiv 0$. Then we use the previous proof on $R(\beta_j + \frac{1}{\zeta})$ and we still get the representation (14).

▶ For theorem 1 on page 29, we have the following stronger version:

Theorem 1 (Stronger version) *The smallest convex set which contains all the zeros of $P(z)$ also contains the zeros of $P'(z)$.*

Proof: Let $\alpha_1, \cdots, \alpha_n$ be the zeros of P, so

$$P(z) = a_n(z - \alpha_1) \cdots (z - \alpha_n).$$

Then

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \cdots + \frac{1}{\alpha_n}.$$

If z_0 is a zero of $P'(z)$ and $z_0 \neq$ each α_i, then this vanishes for $z = z_0$; conjugating the equation gives

$$\frac{z_0 - \alpha_1}{|z_0 - \alpha_1|^2} + \cdots + \frac{z_0 - \alpha_n}{|z_0 - \alpha_n|^2} = 0,$$

so

$$z_0 = m_1\alpha_1 + \cdots + m_n\alpha_n,$$

where

$$m_i \geq 0 \text{ and } \sum_{i=1}^{n} m_i = 1.$$

We now only need to prove the following simple result:

Proposition 1 *Given* $a_1, \cdots, a_n \in \mathbb{C}$*, the set*

$$\{\sum_{i=1}^{n} m_i a_i \mid m_i \geq 0,\ \sum_{i=1}^{n} m_i = 1\} \tag{1}$$

is the intersection C *of all convex sets containing all* a_i *(which is called the convex hull of* $a_1, \cdots, a_n$*).*

Proof: We must show that each point $\sum_{i=1}^{n} a_i m_i$ in (1) is contained in each convex set containing the a_i and thus in C. We may assume it has the form

$$x = \sum_{i=1}^{p} m_i a_i$$

where

$$m_i > 0 \ \text{ for } 1 \leq i \leq p$$

and

$$m_j = 0 \ \text{ for } j > p.$$

We prove $x \in C$ by induction on p. Statement is clear if $p = 1$. Put

$$\lambda = \sum_{i=1}^{p-1} m_i$$

and

$$a = \sum_{i=1}^{p-1} \frac{m_1}{\lambda} a_i.$$

By inductive assumption, $a \in C$. But

$$x = \sum_{i=1}^{p} m_i a_i = \lambda a + (1 - \lambda) a_i$$

where $0 \leq \lambda \leq 1$. So $x \in C$ as stated. **Q.E.D.**

Solution to 4 on p.33

Suppose $R(z)$ is rational and

$$|R(z)| = 1$$

for $|z| = 1$. Then

$$|R(e^{i\theta})| \equiv 1 \qquad \theta \in \mathbb{R}.$$

Let $S(z)$ be the rational functions obtained by conjugating all the coefficients in $R(z)$, then

$$R(e^{i\theta})S(e^{-i\theta}) = R(e^{i\theta})\overline{R(e^{i\theta})} = 1.$$

So

$$R(z)S(\frac{1}{z}) = 1 \text{ on } |z| = 1.$$

Clearing denominators we see this relation

$$R(z)S(\frac{1}{z}) = 1$$

holds for all $z \in \mathbb{C}$.

Since a polynomial has only finitely many zeroes, let

$$\alpha_1, \cdots, \alpha_p$$

be all the zeroes of $R(z)$ which are not equal to 0 or ∞. Then

$$\frac{1}{\alpha_1}, \cdots, \frac{1}{\alpha_p}$$

are the poles of $S(z)$ which are not equal to 0 or ∞. So

$$\frac{1}{\bar{\alpha}_1}, \cdots, \frac{1}{\bar{\alpha}_p}$$

are the poles of $R(z)$ which are not equal to 0 or ∞ because of the definition of S. Then

$$R(z)\left(\frac{z-\alpha_1}{1-\bar{\alpha}_1 z}\cdots\frac{z-\alpha_p}{1-\bar{\alpha}_p z}\right)^{-1}$$

has no poles or zeros except possibly 0 and ∞. Hence

$$R(z) = Cz^l \frac{z-\alpha_1}{1-\bar{\alpha}_1 z}\cdots\frac{z-\alpha_p}{1-\bar{\alpha}_p z}$$

where C is constant with $|C| = 1$, l is integer.

Conversely, such R has $|R(z)| = 1$ on $|z| = 1$.