Stirling’s Formula: an Approximation of the Factorial

Eric Gilbertson

Outline

- Introduction of formula
- Convex and log convex functions
- The gamma function
- Stirling's formula

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Introduction of Formula

In the early 18th century James Stirling proved the following formula:

$$n! = \sqrt{2\pi n}\, n^{n+1/2} e^{-n+\theta/12n}$$

For some $0 < \theta < 1$

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For some $0 < \theta < 1$

This means that as $n \to \infty$

$$n! \sim \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

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Convex Functions

A function f(x) is called *convex* on the interval (a,b) if the function

$$\phi(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

Is a monotonically increasing function of x_1 on the interval

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Is a monotonically increasing function of x_1 on the interval

Ex: x^2 is convex

x^3 is not convex on the interval (-1,0)

Convex Functions - Properties

If $f(x)$ and $g(x)$ are convex then $f(x)+g(x)$ is also convex

If $f(x)$ is twice differentiable and the second derivative of f is positive on an interval, then $f(x)$ is convex on the interval

Log Convex Functions

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Ex: e^{x^2} is log convex

Log Convex Functions - Properties

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If f(t,x) is a log convex function twice differentiable in x, for t in the interval [a,b] and x in any interval then

$$\int_a^b f(t,x)dt$$ is a log convex function of x

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Ex: $$\int_a^b e^{-t} t^{x-1} dt$$ is log convex

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The Gamma Function

An extension of the factorial to all positive real numbers is the gamma function where

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Using integration by parts, for integer n

$$\Gamma(n) = (n-1)!$$

The Gamma Function - Properties

- The Gamma function is log convex

the integrand is twice differentiable on $[0,\infty)$

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the integrand is twice differentiable on $[0,\infty)$

- And $\Gamma(1)=\int_0^\infty e^{-t}dt=1$

The Gamma Function - Uniqueness

Theorem: If a function f(x) satisfies the following three conditions then it is identical to the gamma function.

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Theorem: If a function f(x) satisfies the following three conditions then it is identical to the gamma function.

(1) $f(x+1) = f(x)$

(2) The domain of f contains all $x>0$ and f(x) is log convex

(3) $f(1)=1$

The Gamma Function - Uniqueness

Proof: Suppose $f(x)$ satisfies the three properties. Then since $f(1)=1$ and $f(x+1)=xf(x)$, for integer $n\geq 2$,

$$f(x+n)=(x+n-1)(x+n-2)\ldots(x+1)xf(x)$$

$$f(n) = (n-1)!$$

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Now if we can show $\Gamma(x)$ and $f(x)$ agree on [0,1], then by these properties they agree everywhere.

The Gamma Function - Uniqueness

By property (2) f(x) is log convex, so by definition of convexity

$$\frac{\ln f(-1+n)-\ln f(n)}{(-1+n)-n} \leq \frac{\ln f(x+n)-\ln f(n)}{(x+n)-n} \leq \frac{\ln f(1+n)-\ln f(n)}{(1+n)-n}$$

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$$\ln(n-1) \le \frac{\ln f(x+n)-\ln(n-1)!}{x} \le \ln n$$

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$$\ln(n-1) \le \frac{\ln f(x+n)-\ln(n-1)!}{x} \le \ln n$$

$$\ln[(n-1)^x(n-1)!] \le \ln f(x+n) \le \ln[n^x(n-1)!]$$

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$$\ln[(n-1)^x (n-1)!] \le \ln f(x+n) \le \ln[n^x (n-1)!]$$

The logarithm is monotonic so

$$(n-1)^x (n-1)! \le f(x+n) \le n^x (n-1)!$$

The Gamma Function - Uniqueness

Since f(x+n)=x(x+1)…(x+n-1)f(x) then

$$(n-1)^x (n-1)! \le x(x+1)...(x+n-1) f(x) \le n^x (n-1)!$$

The Gamma Function - Uniqueness

Since f(x+n)=x(x+1)…(x+n-1)f(x) then

$$(n-1)^x (n-1)! \le x(x+1)...(x+n-1)\,f(x) \le n^x (n-1)!$$

$$\frac{(n-1)^x (n-1)!}{x(x+1)...(x+n-1)} \le f(x) \le \frac{n^x (n-1)!}{x(x+1)...(x+n-1)} = \frac{n^x n!(x+n)}{x...(x+n)n}$$

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Since this holds for n ≥ 2, we can replace n by n+1 on the right. Then

$$\frac{n^x n!}{x(x+1)...(x+n)} \leq f(x) \leq \frac{n^x n! (x+n)}{x(x+1)...(x+n)n}$$

The Gamma Function - Uniqueness

This simplifies to

$$f(x)\frac{n}{x+n} \le \frac{n^x n!}{x(x+1)\ldots(x+n)} \le f(x)$$

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$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\ldots(x+n)}$$

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As n goes to infinity,

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\ldots(x+n)}$$

Since Gamma(x) satisfies the 3 properties of the theorem, then it satisfies this limit also.

Thus $\Gamma(x) = f(x)$

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Stirling's Formula

Goal: Find upper and lower bounds for $\Gamma(x)$

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From the definition of e, for k=1,2,…,(n-1)

$$(1+1/k)^k \le e \le (1+1/k)^{k+1}$$

Stirling's Formulas

Goal: Find upper and lower bounds for Gamma(x)

From the definition of e, for k=1,2,…,(n-1)

$$(1+1/k)^k \le e \le (1+1/k)^{k+1}$$

Multiply all of these together to get

$$(\frac{n}{n-1})^{n-1}(\frac{n-1}{n-2})^{n-2}\ldots \le e^{n-1} \le (\frac{n}{n-1})^{n}(\frac{n-1}{n-2})^{n-1}\ldots$$

Stirling's Formula

Simplifying,

$$en^n e^{-n} \leq n! \leq en^{n+1} e^{-n}$$

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$$en^n e^{-n} \le n! \le en^{n+1} e^{-n}$$

$$\Rightarrow en^n e^{-n} \le \Gamma(n+1) \le en^{n+1} e^{-n}$$

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Simplifying,

$$en^n e^{-n} \le n! \le en^{n+1} e^{-n}$$

$$\Rightarrow en^n e^{-n} \le \Gamma(n+1) \le en^{n+1} e^{-n}$$

Guess a function to approximate Γ(x)

$$f(x) = x^{x-1/2} e^{-x} e^{\mu(x)}$$

Stirling's Formula

$\mu(x)$ must be chosen so that

(1) $f(x+1)=xf(x)$

(2) $f(x)$ is log convex

(3) $f(1)=1$

Stirling's Formula

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(1) Is equivalent to $f(x+1)/f(x)=x$

Stirling's Formulas

Simplifying f(x+1)/f(x) gives

$$(1+1/x)^{x+1/2}e^{-1}e^{\mu(x+1)-\mu(x)} = x$$

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$$\Rightarrow \mu(x+1)-\mu(x) = (x+1/2)\ln(1+1/x)-1 \equiv g(x)$$

Then $\mu(x) = \sum_{n=0}^{\infty} g(x+n)$ satisfies the equation

$$\sum_{n=0}^{\infty} g(x+n+1) - \sum_{n=0}^{\infty} g(x+n) = g(x)$$

Stirling's Formula

Consider the Taylor expansion

$$1/2\ln[\frac{1+y}{1-y}] = \frac{y}{1} + \frac{y^3}{3} + \frac{y^5}{5} + \ldots \text{ with } \quad y = \frac{1}{2x+1}$$

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$$\Rightarrow 1/2\ln(1+1/x) = \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \ldots$$

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$$\Rightarrow (x+1/2)\ln(1+1/x) - 1 = g(x) = \frac{1}{3(2x+1)^2} + \frac{1}{5(2x+1)^4} + \ldots$$

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$$\Rightarrow (x+1/2)\ln(1+1/x) - 1 = g(x) = \frac{1}{3(2x+1)^2} + \frac{1}{5(2x+1)^4} + \ldots$$

Replacing the denominators with all 3's gives

$$g(x) \le \frac{1}{3(2x+1)^2} + \frac{1}{3(2x+1)^4} + \ldots$$

Stirling's Formula

g(x) is a geometric series, thus

$$g(x) \le \frac{1}{3(2x+1)^2} \frac{1}{1 - \frac{1}{(2x+1)^2}} = \frac{1}{12x} - \frac{1}{12(x+1)}$$

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$$\mu(x) = \sum_{n=0}^{\infty} g(x+n) \le \sum_{n=0}^{\infty} \left(\frac{1}{12(x+n)} - \frac{1}{12(x+n+1)}\right)$$

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$$= \frac{1}{12x}$$

Stirling's Formula

Since g(x) is positive then $0 < \mu(x) < \dfrac{1}{12x}$

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For property (2) we need to show f(x) is log convex

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For property (2) we need to show f(x) is log convex

This means $\ln f(x) = (x - \frac{1}{2}) \ln x - x + \mu(x)$

must be convex.

Stirling's Formula

$\mu(x)$ is convex because g(x) is the sum of convex functions

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derivative $\frac{1}{x}+\frac{1}{2x^2}$ is positive for positive x.

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Thus f(x) is log convex.

Stirling's Formulas

For property (3) we need f(1)=1 so that

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Consider the function

$$h(x) = 2^x\Gamma(x/2)\Gamma(\frac{x+1}{2})$$

Stirling's Formulas

For property (3) we need f(1)=1 so that

$$\Gamma(x) = ax^{x-1/2}e^{-x+\theta/12x}$$

Consider the function

$$h(x) = 2^x\Gamma(x/2)\Gamma(\frac{x+1}{2})$$

h(x) is log convex since the second derivative of $\ln 2^x$ is nonnegative and the gamma function is log convex.

Stirling's Formula

$$h(x+1) = 2^{x+1}\Gamma(\frac{x+1}{2})\Gamma(x/2+1)$$

Stirling's Formula

$$h(x+1) = 2^{x+1}\Gamma(\frac{x+1}{2})\Gamma(x/2+1)$$

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$$h(x+1) = 2^{x+1}\Gamma(\frac{x+1}{2})\Gamma(x/2+1)$$

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$$\Rightarrow h(x+1) = xh(x)$$

Stirling's Formula

$$h(x+1) = 2^{x+1}\Gamma(\frac{x+1}{2})\Gamma(x/2+1)$$

$$= 2\frac{x}{2}2^{x}\Gamma(x/2)\Gamma(\frac{x+1}{2})$$

$$\Rightarrow h(x+1) = xh(x)$$

This means h(x) satisfies properties (1) and (2) of the uniqueness theorem, thus

$$h(x) = a_2\Gamma(x) \qquad \text{for some constant } a_2$$

Stirling's Formula

Setting x=1 gives

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$$h(1) = 2\Gamma(1/2)\Gamma(1) = a_2\Gamma(1)$$
$$\Rightarrow a_2 = 2\Gamma(1/2)\Gamma(1)$$

Using the limit equation for the gamma function

$$2\Gamma(1/2)\Gamma(1) = 2\lim_{n\to\infty}\frac{n^{3/2}n!^2 2^{n+1}}{(2n+2)!}$$

Stirling's Formula

$$a_2 = 2\lim_{n\to\infty}\frac{(n!)^2\,2^{2n}}{(2n)!\,n^{1/2}}$$

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$$= 2\lim_{n\to\infty}\frac{[a^2 n^{2n+1} e^{-2n} e^{2\theta_1/12n}]2^{2n}}{[a(2n)^{2n+1/2} e^{-2n} e^{\theta_2/24n}]n^{1/2}}$$

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$$= a\sqrt{2}\lim_{n\to\infty} e^{2\theta_1/12n-\theta_2/24n}$$

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$$= a\sqrt{2}\lim_{n\to\infty} e^{2\theta_1/12n-\theta_2/24n}$$

$$\Rightarrow a_2 = 2\Gamma(1/2) = a\sqrt{2}$$

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$$a_2 = 2\lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n)!\,n^{1/2}}$$

$$= 2\lim_{n\to\infty}\frac{[a^2 n^{2n+1} e^{-2n} e^{2\theta_1/12n}]2^{2n}}{[a(2n)^{2n+1/2} e^{-2n} e^{\theta_2/24n}]n^{1/2}}$$

$$= a\sqrt{2}\lim_{n\to\infty} e^{2\theta_1/12n-\theta_2/24n}$$

$$\Rightarrow a_2 = 2\Gamma(1/2) = a\sqrt{2} \qquad \Rightarrow a = \sqrt{2}\Gamma(1/2) = \sqrt{2\pi}$$

Stirling's Formula

Substituting a in the formula for f gives Stirling's final result:

$$n! = \sqrt{2\pi n}\, n^{n+1/2} e^{-n+\theta/12n}$$

for some $0 < \theta < 1$