Generating Functions and Their Applications

Agustinus Peter Sahanggamu
MIT Mathematics Department
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Abstract. Generating functions have useful applications in many fields of study. In this paper, the generating functions will be introduced and their applications in combinatorial problems, recurrence equations, and physics will be illustrated.

1. Introduction.

Working with a continuous function is sometimes much easier than working with a sequence. For example, in the analysis of functions, calculus is very useful. However, the discrete nature of sequences prevents us from using calculus on sequences. A generating function is a continuous function associated with a given sequence. For this reason, generating functions are very useful in analyzing discrete problems involving sequences of numbers or sequences of functions.

Definition 1-1. The generating function of a sequence $\{f_n\}_{n=0}^{\infty}$ is defined as

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \tag{1-1}$$

for $|x| < R$, and R is the radius of convergence of the series.

It is important that the series has a nonzero radius of convergence, otherwise $f(x)$ would be undefined for all $x \neq 0$. Fortunately, for most sequences that we would be dealing with, the radius of convergence is positive. However, we can certainly construct sequences for which the series (1-1) is divergent for all $x \neq 0$; $f_n = n^n$ is one such example.

Example 1-2. As a simple example of a generating function, consider a geometric sequence, $f_n = a^n$. Then

$$f(x) = \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1 - ax}. \tag{1-2}$$

This series convergences absolutely whenever $|ax| < 1$. Therefore, the radius of convergence is $R = 1/|a|$.

For the rest of the paper, if not mentioned otherwise, x is always chosen to be small enough such that any series encountered in our analysis converges absolutely. Now we shall discuss an application of generating functions to linear recurrence problems.

2. From Recursion to Algebra.

Generating functions can be used to solve a linear recurrence problem.

Definition 2-1. The problem of linear recurrence is to find the values of a sequence $\{u_n\}$ satisfying

$$\sum_{k=0}^{m} c_k u_{n+k} = v, \quad \text{for some constant } v \text{ and any integer } n \geq 0, \tag{2-1}$$

given the initial values $u_0, u_1, \ldots, u_{m-1}$, where both c_0 and c_m are nonzero.

In order to solve this recurrence problem, we use the following property of generating functions.

Proposition 2-2. *If $\{u_n\}_{n=0}^{\infty}$ is a sequence with a generating function $u(x)$ and k is a positive integer, then*

$$\sum_{n=0}^{\infty} u_{n+k} x^n = \frac{1}{x^k}\left[u(x) - \sum_{j=0}^{k-1} u_j x^j\right]. \tag{2-2}$$

Proof: We start from the definition of $u(x)$.

$$u(x) = \sum_{j=0}^{\infty} u_j x^j = \sum_{j=0}^{k-1} u_j x^j + \sum_{j=k}^{\infty} u_j x^j. \tag{2-3}$$

Shifting the summation index in the second term to $n = j - k$, we obtain

$$u(x) = \sum_{j=0}^{k-1} u_j x^j + \sum_{n=0}^{\infty} u_{n+k} x^{n+k}, \tag{2-4}$$

from which Eq. (2-2) immediately follows. □

By multiplying both sides of Eq. (2-1) by x^n and summing from $n = 0$ to ∞, we find

$$\sum_{k=0}^{m} c_k \sum_{n=0}^{\infty} u_{n+k} x^n = v \sum_{n=0}^{\infty} x^n = \frac{v}{1-x}. \tag{2-5}$$

Using Proposition 2-2, each term on the left hand side with fixed k can be expressed in terms of $u(x)$ and known constants $u_0, u_1, \ldots, u_{m-1}$. We have now reduced Eq. (2-5) into an algebraic equation for $u(x)$, which can be easily solved.

After finding $u(x)$, we write down the Taylor series expansion of $u(x)$ around $x = 0$. Because the Taylor series of a function is unique (if it exists), the coefficient of x^n in the Taylor series must be u_n. To illustrate this method, we shall use it on the following example from physics.

Example 2-3. In special relativity, the usual one dimensional velocity addition formula $v' = u + v$ is modified into [**1**, p. 127]

$$v' = \frac{u+v}{1+uv}, \tag{2-6}$$

with v', u, and v measured in units of the speed of light c. We will use this velocity addition in the following problem. Suppose there are infinitely many cars labeled by integers $n \geq 0$. The $(n+1)$-th car moves to the right relative to the n-th car with a relative velocity v $(0 < v < 1)$. In our reference frame, we denote the velocity of the n-th car by u_n. Assuming $u_0 = 0$, find u_n for all $n \geq 1$.

Solution: Notice that u_{n+1} is the addition of u_n and v using the addition formula in Eq. (2-6).

$$u_{n+1} = \frac{u_n + v}{1 + u_n v}, \quad \text{for } n \geq 0. \tag{2-7}$$

This recurrence is not linear, and therefore we may not apply our previous method directly. With a little manipulation, however, this recurrence can be transformed into a linear recurrence.

$$\begin{aligned} 1 - u_{n+1} &= \frac{1 + u_n v - u_n - v}{1 + u_n v} = \frac{(1-u_n)(1-v)}{1 + v - v(1-u_n)} \\ \frac{1}{1-u_{n+1}} &= \left(\frac{1+v}{1-v}\right)\frac{1}{1-u_n} - \frac{v}{1-v}. \end{aligned} \tag{2-8}$$

Defining

$$\alpha = \frac{1+v}{1-v}, \ \lambda = \frac{v}{1-v}, \text{ and } f_n = \frac{1}{1-u_n},$$

Eq. (2-8) can be written as

$$f_{n+1} = \alpha f_n - \lambda, \tag{2-9}$$

which is a linear recurrence in f_n. Now multiply both sides by x^n and sum from $n = 0$ to ∞.

$$\begin{aligned} \sum_{n=0}^{\infty} f_{n+1} x^n &= \alpha \sum_{n=0}^{\infty} f_n x^n - \lambda \sum_{n=0}^{\infty} x^n \\ \frac{1}{x}\big(f(x) - f_0\big) &= \alpha f(x) - \frac{\lambda}{1-x}, \end{aligned} \tag{2-10}$$

where we have used Proposition 2-2 to simplify the left hand side. The initial condition of f_n is given by $f_0 = 1/(1-u_0) = 1$. Solving for $f(x)$ yields

$$\begin{aligned} f(x) &= \frac{1}{1-\alpha x} - \frac{\lambda x}{(1-x)(1-\alpha x)} \\ f(x) &= \frac{1}{1-\alpha x} + \frac{\lambda}{\alpha - 1}\left[\frac{1}{1-x} - \frac{1}{1-\alpha x}\right]. \end{aligned} \tag{2-11}$$

Using the definitions of α and λ, we find

$$\frac{\lambda}{\alpha - 1} = \frac{1}{2}, \tag{2-12}$$

and hence

$$f(x) = \frac{1}{2}\left[\frac{1}{1-\alpha x} + \frac{1}{1-x}\right] = \sum_{n=0}^{\infty} \frac{1}{2}\big(\alpha^n + 1\big) x^n = \sum_{n=0}^{\infty} f_n x^n. \tag{2-13}$$

Since $f(x)$ is the sum of two geometric series, we conclude that the Taylor series around 0 has a positive radius of convergence. Therefore, $2f_n = \alpha^n + 1$ by the uniqueness of Taylor series, and

$$u_n = 1 - \frac{1}{f_n} = \frac{\alpha^n - 1}{\alpha^n + 1}. \tag{2-14}$$

Since $\alpha > 1$, we conclude that $0 < u_n < 1$ for all $n \geq 1$. Physically, this result shows that any car moves with a speed less than c (remember that we are writing u_n in units of the speed of light). Special relativity predicts that any massive object always travels slower than light [**4**, p. 119]. □

3. Applications to Combinatorial Problems.

Many combinatorial problems can be solved with the aid of generating functions. In particular, let's consider the problem of finding the number of partitions of a natural number.

Definition 3-1. [**6**, p. 169] A partition of a natural number n is a way to write n as a sum of natural numbers, without regard to the ordering of the numbers.

Example 3-2. $1+1+3+1$ is a partition of 6.

With this definition, the generating function of the number of partitions of n has a simple form.

Theorem 3-3. [**6**, p. 169] *If p_n is the number of partitions of n and $p_0 = 1$, then*

$$\sum_{n=0}^{\infty} p_n x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{3-1}$$

Proof: First we need to establish the convergence of the infinite product for $x \neq 0$. This infinite product converges absolutely if the series $\displaystyle\sum_{k=1}^{\infty} x^k$ converges absolutely [**3**, p. 53]. Thus, the right hand side converges absolutely for $|x| < 1$.

Each factor in the infinite product can be expressed as a geometric series.

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \left[\sum_{m=0}^{\infty} x^{mk} \right]. \tag{3-2}$$

In this form, we can see that the coefficient of the x^n term is equal to the number of ways to choose integers $\{m_k | m_k \geq 0,\, k = \text{positive integers}\}$ satisfying $n = \displaystyle\sum_{k=1}^{\infty} m_k k$. If we take the $x^{m_k k}$ term from the k-th factor, then we obtain x^n. By comparing this with Definition 3-1, we conclude that the coefficient of x^n is equal to the number of partitions of n. □

As a check, let's try expanding the right hand side of Eq. (3-1) up to x^4.

$$\begin{aligned}
\sum_{n=0}^{\infty} p_n x^n &= (1+x+x^2+x^3+x^4)(1+x^2+x^4)(1+x^3)(1+x^4) + \cdots \\
&= (1+x+x^2+x^3+x^4)(1+x^2+x^3+2x^4) + \cdots \\
&= 1 + x + 2x^2 + 3x^3 + 5x^4 + \mathcal{O}(x^5),
\end{aligned}$$

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 3, \quad p_4 = 5. \tag{3-3}$$

We can easily verify that Eq. (3-3) correctly gives the number of partitions of 1 to 4.

Another important combinatorial problem that can be easily solved with generating functions is Catalan's problem [**6**, p. 260].

Example 3-4 (Catalan's Problem). Given a product of n letters, how many ways can we calculate the product by multiplying two factors at a time, keeping the order fixed? As an example, for $n = 3$, there are two ways: $(a_1a_2)a_3$ and $a_1(a_2a_3)$.

Solution: Denote the solution for $n = m$ by K_m. It is clear that $K_2 = 1$. For later convenience, we define $K_1 = 1$. Now consider the $n = m + 1$ case. Suppose at the last step of the multiplication, we have bc, where $b = a_1 \cdots a_j$, $c = a_{j+1} \cdots a_{m+1}$, and $1 \le j \le m$. Notice that there are K_j ways to multiply the factors in b, and there are K_{m-j+1} ways to multiply the factors in c. Thus, for a given j, there are K_jK_{m-j+1} ways to multiply $a_1 \cdots a_{m+1}$. The total number of ways can be obtained by summing over all possible values of j.

$$K_{m+1} = \sum_{j=1}^{m} K_j K_{m-j+1}, \quad m \ge 1. \tag{3-4}$$

Multiply both sides by x^{m+1} and sum from $m = 1$ to $m = \infty$.

$$\sum_{m=1}^{\infty} K_{m+1}x^{m+1} = \sum_{m=1}^{\infty}\sum_{j=1}^{m} K_j K_{m-j+1} x^{m+1}. \tag{3-5}$$

As usual, we define the generating function for $\{K_n\}_{n=1}^{\infty}$.

$$K(x) = \sum_{n=1}^{\infty} K_n x^n. \tag{3-6}$$

The left hand side of Eq. (3-5) is

$$\sum_{m=1}^{\infty} K_{m+1}x^{m+1} = \sum_{n=1}^{\infty} K_n x^n - K_1 x = K(x) - x. \tag{3-7}$$

Now consider the expression for $K(x)^2$.

$$K(x)^2 = \sum_{j=1}^{\infty} K_j x^j \sum_{i=1}^{\infty} K_i x^i. \tag{3-8}$$

Let $i = m - j + 1$, where $m \ge 1$. For a given m, j can be any integer from 1 to m, since $i \ge 1$. Thus, we can rewrite Eq. (3-8) as

$$K(x)^2 = \sum_{m=1}^{\infty}\sum_{j=1}^{m} K_j K_{m-j+1} x^j x^{m-j+1} = \sum_{m=1}^{\infty}\sum_{j=1}^{m} K_j K_{m-j+1} x^{m+1}. \tag{3-9}$$

By using Eqs. (3-7) and (3-9) in Eq. (3-5), we obtain a quadratic equation for $K(x)$.

$$K(x)^2 - K(x) + x = 0, \quad \text{or} \quad K(x) = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-4x}. \tag{3-10}$$

Notice from Eq. (3-6) that $K(0) = 0$, which means we must take the negative sign for the square root.

$$K(x) = \frac{1 - \sqrt{1-4x}}{2}. \tag{3-11}$$

It is clear that the square root has a converging power series around $x=0$ for $|x|<\frac{1}{4}$, and hence the infinite series defining $K(x)$ has a radius of convergence of $\frac{1}{4}$.

Use the binomial formula to obtain the power series expansion of $\sqrt{1-4x}$.

$$\begin{aligned}\sqrt{1-4x} &= 1+\sum_{n=1}^{\infty}\frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}(-4x)^n\\ &= 1-2x+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}(-1)^n 2^n x^n\\ &= 1-2x-2\sum_{n=2}^{\infty}\frac{1\cdot 2\cdot 3\cdots(2n-2)}{n!(n-1)!}x^n\\ &= 1-2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2(n-1)}{n-1}x^n. \qquad (3\text{-}12)\end{aligned}$$

Now we substitute Eq. (3-12) into Eq. (3-11) to find

$$\sum_{n=1} K_n x^n = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2(n-1)}{n-1}x^n, \qquad (3\text{-}13)$$

or

$$K_{m+1}=\frac{1}{m+1}\binom{2m}{m}, \quad m\geq 0. \qquad (3\text{-}14)$$

This problem was solved by Catalan in 1838 [**6**, p. 259-260], and the Catalan numbers are conventionally defined as $C_n = K_{n+1}$, for $n\geq 0$. □

There are many other applications of generating functions in combinatorial problems that cannot be covered here. A wide variety of examples are discussed in [**5**, Chapter 3].

4. Legendre Polynomials.

So far, we have only discussed generating functions of sequences of numbers. However, in Section 1, I mentioned that generating function methods can also be used to analyze sequences of *functions*. One interesting example is the generating function of Legendre polynomials. As we shall see, the generating function provides a physical insight, with a deep connection to electromagnetism.

There are several ways to define the Legendre polynomials $P_n(t)$. For example, they can be defined as solutions to a differential equation [**2**, p. 96]. For our purposes, however, it is more convenient to define $P_n(t)$ as follows.

Definition 4-1. The Legendre polynomials $\{P_n(t)\}_{n=0}^{\infty}$ are defined in the interval $-1\leq t\leq 1$. [**2**, p. 100] They satisfy the recurrence relation

$$(n+1)P_{n+1}(t)=(2n+1)tP_n(t)-nP_{n-1}(t) \quad \text{for } n\geq 1, \qquad (4\text{-}1)$$

with $P_0(t)=1$ and $P_1(t)=t$.

From this definition, it is easy to prove by induction that $P_n(t)$ is a polynomial in t of degree n. We now want to find the generating function of $P_n(t)$. In order to avoid confusion in the notation, we denote the generating function of $P_n(t)$ for *fixed* t as

$$Q_t(x)=\sum_{n=0}^{\infty}P_n(t)x^n. \qquad (4\text{-}2)$$

Taking the derivative gives

$$xQ_t'(x) = \sum_{n=1}^{\infty} nP_n(t)x^n. \tag{4-3}$$

Let's multiply Eq. (4-1) by x^{n+1} and sum from $n = 1$ to $n = \infty$.

$$\sum_{n=1}^{\infty}(n+1)P_{n+1}(t)x^{n+1} = \sum_{n=1}^{\infty}\big[(2n+1)tP_n(t) - nP_{n-1}(t)\big]x^{n+1}$$

$$\sum_{m=2}^{\infty} mP_m(t)x^m = 2tx\sum_{n=1}^{\infty} nP_n(t)x^n + x^2\sum_{k=0}^{\infty}\big[tP_{k+1}(t)x^k - (k+1)P_k(t)x^k\big]. \tag{4-4}$$

The last step follows from the substitutions $m = n+1$ and $k = n-1$.

We note from Eqs. (4-2) and (4-3) that

$$\sum_{m=2}^{\infty} mP_m(t)x^m = xQ_t'(x) - P_1(t)x = xQ_t'(x) - tx, \tag{4-5}$$

$$\sum_{k=0}^{\infty}(k+1)P_k(t)x^k = xQ_t'(x) + Q_t(x), \tag{4-6}$$

while Proposition 2-2 implies

$$x\sum_{k=0}^{\infty} P_{k+1}(t)x^k = Q_t(x) - P_0(t) = Q_t(x) - 1. \tag{4-7}$$

Thus, Eq. (4-4) simplifies to

$$xQ_t'(x) - tx = 2tx^2Q_t'(x) + tx\big(Q_t(x) - 1\big) - x^2\big(xQ_t'(x) + Q_t(x)\big),$$

$$Q_t'(x) = -\frac{x-t}{1-2tx+x^2}Q_t(x). \tag{4-8}$$

By integrating Eq. (4-8) and imposing the initial condition $Q_t(0) = P_0(t) = 1$, we obtain

$$Q_t(x) = \sum_{n=0}^{\infty} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}. \tag{4-9}$$

To find the radius of convergence of the power series of $Q_t(x)$, we need to find the location z_s (in complex plane) of the singularity nearest to the origin. $Q_t(z_s)$ is singular if

$$1 - 2tz_s + z_s^2 = 0, \tag{4-10}$$

$$z_s = t \pm i\sqrt{1-t^2}, \quad |z_s| = 1. \tag{4-11}$$

Therefore, $Q(z)$ is analytic in the region $|z| < 1$, and its power series converges absolutely in this region.

In electrostatics, the potential along the z axis due to an azimuthally symmetric volume charge distribution $\rho(r,\theta)$ is given by [**2**, p. 35] (we set $4\pi\epsilon_{\text{o}} = 1$)

$$V(z) = 2\pi\int_0^{\infty} dr'\, r'^2 \int_0^{\pi} d\theta'\, \sin\theta' \frac{\rho(r',\theta')}{\sqrt{z^2 - 2zr'\cos\theta' + r'^2}}. \tag{4-12}$$

If $\rho(r', \theta')$ is bounded, $\rho(r', \theta') = 0$ for $r' > a$, and we are only interested in $V(z)$ for $z > a$, then

$$V(z) = 2\pi \int_0^a dr' \, \frac{r'^2}{z} \int_0^\pi d\theta' \sin\theta' \frac{\rho(r', \theta')}{\sqrt{1 - 2x\cos\theta' + x^2}}, \tag{4-13}$$

where $x = r'/z < 1$.

Now we may use Eq. (4-9) with $t = \cos\theta'$ because $|\cos\theta'| \leq 1$. Eq. (4-13) can then be written as

$$V(z) = \sum_{n=0}^{\infty} \frac{q_n}{z^{n+1}}, \tag{4-14}$$

with

$$q_n = 2\pi \int_0^a dr' (r')^{n+2} \int_0^\pi d\theta' \, \sin\theta' \rho(r', \theta') P_n(\cos\theta')$$

$$q_n = \int d^3r' \, r'^n \rho(r', \theta') P_n(\cos\theta'). \tag{4-15}$$

The numbers q_n are called the multipole moments. In particular, q_0 is the monopole moment (or total charge), and q_1 is the dipole moment [**2**, p. 146]. Since the n-th moment term in the potential falls off as $1/z^{n+1}$, the first nonzero moment q_n characterizes the behavior of $V(z)$ as $z/a \to \infty$.

We can see that the generating function of $P_n(t)$ appears naturally in electromagnetism. This technique of expanding the potential as a series of "moments" is very useful, and is called "multipole expansion".

An important property of $P_n(t)$ can be shown directly from the generating function by considering $Q_t(-x)$.

$$Q_t(-x) = \sum_{n=0}^{\infty} P_n(t)(-x)^n = \frac{1}{\sqrt{1 + 2tx + x^2}}. \tag{4-16}$$

Notice that the right hand side is also equivalent to $Q_{-t}(x)$.

$$\sum_{n=0}^{\infty} (-1)^n P_n(t) x^n = \sum_{n=0}^{\infty} P_n(-t) x^n. \tag{4-17}$$

From the uniqueness of the power series of $Q_t(-x)$, we obtain $P_n(-t) = (-1)^n P_n(t)$. Therefore $P_n(t)$ is an odd (even) polynomial if and only if n is odd (even).

Another property can be obtained by setting $t = 1$ in Eq. (4-9).

$$\sum_{n=0}^{\infty} P_n(1) x^n = \frac{1}{\sqrt{1 - 2x + x^2}} = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n. \tag{4-18}$$

Thus, $P_n(1) = 1$ for all n. Since $P_n(t)$ is odd for odd n, we also obtain $P_n(-1) = (-1)^n$.

5. Useful Trick to Find a Generating Function.

In Section 2, we saw that we can easily find the generating function of a sequence if that sequence is defined through a *linear* recurrence. However, in some cases, we may not have a linear recurrence, such as in the Catalan's problem in Section 3. For some sequences without a linear recurrence, it is possible to obtain the generating function using a convolution property. In fact, we have actually used this property to solve the Catalan's problem.

Definition 5-1. A convolution of two sequences $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ is another sequence denoted by $\{(f*g)_n\}_{n=0}^{\infty}$, with

$$(f*g)_n = \sum_{k=0}^{n} f_k g_{n-k}. \tag{5-1}$$

Theorem 5-2. *Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two sequences with generating functions $u(x)$ and $v(x)$ respectively. If $w_n = (u*v)_n$ and $w(x)$ is the generating function of $\{w_n\}_{n=0}^{\infty}$, then*

$$w(x) = u(x)v(x). \tag{5-2}$$

The radius of convergence of $w(x)$ is the minimum of the radii of convergence of $u(x)$ and $v(x)$.

Proof: Let $r > 0$ and $s > 0$ be the radii of convergence of $u(x)$ and $v(x)$. Denote $t = \min(r, s)$. Consider the product

$$u(x)v(x) = \sum_{i=0}^{\infty} u_i x^i \sum_{j=0}^{\infty} v_j x^j, \tag{5-3}$$

for $|x| < t$. Since both series converge absolutely, we may rearrange the terms in the double summation. Suppose we want to group the same powers of x. We can do this by writing $j = n - i$, with $n \geq 0$. For each n, i goes from 0 to n, because j is nonnegative.

$$u(x)v(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} u_i x^i v_{n-i} x^{n-i} = \sum_{n=0}^{\infty} w_n x^n. \tag{5-4}$$

This is precisely the definition of $w(x)$, and the series on the right hand side converges absolutely for $|x| < t$. $\square$

As we shall see later in this section, it is sometimes more convenient to find a generating function for $\{a_n/n!\}$ instead of $\{a_n\}$. This is the motivation to define the *exponential generating function*.

Definition 5-3. The exponential generating function $F(x)$ of a sequence $\{f_n\}_{n=0}^{\infty}$ is defined as

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}. \tag{5-5}$$

The exponential generating functions have the following property.

Lemma 5-4. *If $F(x)$ is the exponential generating function of a sequence $\{f_n\}_{n=0}^{\infty}$, then*

$$\sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!} = F'(x). \tag{5-6}$$

Proof: Differentiate both sides of the definition of $F(x)$ in Eq. (5-5).

$$F'(x) = \sum_{m=1}^{\infty} m f_m \frac{x^{m-1}}{m!} = \sum_{m=1}^{\infty} f_m \frac{x^{m-1}}{(m-1)!} = \sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!}. \tag{5-7}$$

$\square$

We can state a theorem analogous to Theorem 5-2 for exponential generating functions.

Theorem 5-5. *Let* $\{f_n\}_{n=0}^{\infty}$ *and* $\{g_n\}_{n=0}^{\infty}$ *be two sequences with exponential generating functions* $F(x)$ *and* $G(x)$ *respectively. If*

$$h_n = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}, \tag{5-8}$$

and $H(x)$ *is the exponential generating function of* $\{h_n\}_{n=0}^{\infty}$, *then*

$$H(x) = F(x)G(x). \tag{5-9}$$

The radius of convergence of $H(x)$ *is the minimum of the radii of convergence of* $F(x)$ *and* $G(x)$.

Proof: The proof follows the same steps as the proof of Theorem 5-2, by substituting $u_n = f_n/n!$ and $v_n = g_n/n!$. □

We shall now discuss an example to illustrate the convolution method in a problem where the exponential generating function is a more convenient choice.

Example 5-6. (Bell numbers) [**6**, p. 167]. Denote by b_n the number of ways to write a set of n distinct elements as a union of disjoint subsets, with $b_0 = 1$. For $n = 2$, there are two ways: $\{a_1, a_2\}$, and $\{a_1\} \cup \{a_2\}$. Find a formula for b_n.

Solution: First we need to find a recurrence relation for b_n. Consider a set A of $(n+1)$ distinct elements, $A = \{a_1, a_2, \ldots, a_{n+1}\}$. Suppose a_1 is contained in the first subset along with j other elements, where $0 \le j \le n$. There are $\binom{n}{j}$ ways to form this subset, which is the number of ways to pick j elements from $\{a_2, a_3, \cdots, a_{n+1}\}$. Once the first subset is fixed, we are left with a set S containing $(n - j)$ distinct elements. There are b_{n-j} ways of partitioning S into subsets, and therefore we may write

$$b_{n+1} = \sum_{j=0}^{n} \binom{n}{j} b_{n-j} = \sum_{k=0}^{n} \binom{n}{k} b_k, \tag{5-10}$$

by using $k = n - j$ and $\binom{n}{k} = \binom{n}{n-k}$. If we define a sequence $\{t_n = 1\}_{n=0}^{\infty}$, then its exponential generating function $T(x)$ is given by

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \tag{5-11}$$

Eq. (5-10) can be written as

$$w_n = \sum_{k=0}^{n} \binom{n}{k} b_k t_{n-k}, \tag{5-12}$$

with $w_n = b_{n+1}$.

Notice the similarity between Eqs. (5-12) and (5-8). Applying Theorem 5-5 on $\{b_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$, we obtain

$$\begin{aligned} B(x)T(x) &= W(x) \\ B(x)e^x &= \sum_{n=0}^{\infty} w_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} b_{n+1} \frac{x^n}{n!}, \end{aligned} \tag{5-13}$$

where

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \tag{5-14}$$

is the exponential generating function of $\{b_n\}_{n=0}^{\infty}$.

According to Lemma 5-4, the right hand side of Eq. (5-13) can be written as $B'(x)$. Thus,

$$B'(x) = e^x B(x). \tag{5-15}$$

We can integrate Eq. (5-15) and use $B(0) = b_0 = 1$ to find

$$B(x) = e^{e^x - 1} = \frac{1}{e}\, e^{e^x}. \tag{5-16}$$

Since e^x has a power series that converges everywhere, we conclude that $B(x)$ has an infinite radius of convergence. Let's write the power series expansion of $B(x)$.

$$\begin{aligned} B(x) &= \frac{1}{e}\, e^{e^x} = \frac{1}{e}\left[1 + \sum_{k=1}^{\infty} \frac{e^{kx}}{k!}\right] \\ &= \frac{1}{e}\left[1 + \sum_{k=1}^{\infty} \frac{1}{k!}\right] + \frac{1}{e} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k^n}{k!} \frac{x^n}{n!} = 1 + \frac{1}{e} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k^{n-1}}{(k-1)!} \frac{x^n}{n!}. \end{aligned} \tag{5-17}$$

Therefore, for any natural number n,

$$b_n = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^{n-1}}{(k-1)!}. \tag{5-18}$$

To check our answer, take $n = 2$. One way to find b_2 is to sum the infinite series in Eq. (5-18). However, there is a simpler way if we notice that

$$B''(x) = \sum_{n=2}^{\infty} n(n-1) b_n \frac{x^{n-2}}{n!}, \tag{5-19}$$

and thus $b_2 = B''(0)$. We can find $B''(x)$ by differentiating Eq. (5-16) twice.

$$\begin{aligned} B'(x) &= e^x e^{e^x - 1} = e^{e^x + x - 1}, \\ B''(x) &= (1 + e^x) e^{e^x + x - 1}. \end{aligned} \tag{5-20}$$

Therefore, $b_2 = 2$ as expected. Incidentally, by using Eq. (5-18) for $n = 2$, we have proven the following infinite series,

$$\sum_{k=1}^{\infty} \frac{k}{(k-1)!} = 2e. \tag{5-21}$$

$\square$

Conclusions

We have discussed some basic applications of generating functions, as a method to solve a linear recurrence or combinatorial problems. However, there are certainly many more aspects in the subject that are not discussed here. Readers interested to learn more are invited to read [**5**] for a very extensive treatment of generating functions.

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