PRACTICE HOUR TEST (Throughout, μ denotes Lebesgue measure on $\mathbf{R}$.)

1. (20 pts) If J is a finite union of bounded intervals, define $\ell(J)$ as the length of J.

a) Give a definition of Lebesgue outer measure μ^* on $\mathbf{R}$ in terms of ℓ.

b) Prove directly from this definition that $\mu^*([0,1]) = 1$. Use without proof that ℓ is finitely subadditive and finitely additive on finite unions of intervals. You may not, however, use the fact that ℓ is countably subadditive or countably additive.

2. (20 pts) Deduce the dominated convergence theorem from Fatou's lemma. (Your answer must include a careful statement of both the theorem and the lemma.)

3.

a) (10 pts) Show that every Cauchy sequence in $L^1(I,\mu)$ has a subsequence that converges pointwise almost everywhere.

b) (6 pts) Find a Cauchy sequence as in part (a) that does not converge pointwise almost everywhere.

4. (16 pts) Decide if the following statements true or false and give a **reason** if true and a **counterexample** if false. (4 points for each correct answer; 4 points for the reason or counterexample.) As in all parts of the test, μ denotes Lebesgue measure on $\mathbf{R}$.

a) (T/F) If A_k are measurable subsets of $\mathbf{R}$, then $\lim\limits_{N\to\infty} \mu\left(\bigcap\limits_{k=1}^{N} A_k\right) = \mu\left(\bigcap\limits_{k=1}^{\infty} A_k\right)$

b) (T/F) If $f(x,y) \geq 0$ is measurable on $\mathbf{R}\times\mathbf{R}$, and $\int_{\mathbf{R}}\left(\int_{\mathbf{R}} f(x,y)d\mu(x)\right)d\mu(y) < \infty$, then $\dfrac{xyf(x,y)}{x^2+y^2}$ is integrable on $\mathbf{R}\times\mathbf{R}$.

5. (16 pts) Let f_n be a sequence of measurable functions on $[0,1]$ such that $0 \leq f_n(x) \leq 1$. Find the relationship between

$$\limsup_{n\to\infty} \int_0^1 f_n(x)\,d\mu(x) \quad \text{and} \quad \int_0^1 \limsup_{n\to\infty} f_n(x)\,d\mu(x),$$

In other words, decide if they are equal or if one is necessarily less than or equal to the other. Prove your answer and give an example if they can be unequal.

6. (12 pts) Show that for all $f \in L^1(\mathbf{R},\mu)$,

$$\lim_{t\to 0} \int_{\mathbf{R}} |f(x) - f(x+t)|d\mu(x) = 0$$

One way to prove this (*not* the fastest) is to start with 1_E.

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