18.102 Introduction to Functional Analysis
Spring 2009

## Lecture 16. Tuesday, April 7: partially reconstructed

From last time

**Proposition 23.** *The invertible elements form an open subset* $\mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

*Proof.* Recall that we showed using the convergence of the Neumann series that if $B \in \mathcal{B}(\mathcal{H})$ and $\|B\| < 1$ then $\mathrm{Id} - B$ is invertible, meaning it has a two-sided inverse in $\mathcal{B}(\mathcal{H})$ (which we know, from the open mapping Theorem to be equivalent to it being a bijection).

So, suppose $G \in \mathrm{GL}(\mathcal{H})$, meaning it has a two-sided (and unique) inverse $G^{-1} \in \mathcal{B}(\mathcal{H})$ :

$$G^{-1}G = GG^{-1} = \mathrm{Id}\,. \tag{16.1}$$

Then we wish to show that $B(G;\epsilon) \subset \mathrm{GL}(\mathcal{H})$ for some $\epsilon > 0$. In fact we shall see that we can take $\epsilon = \|G^{-1}\|^{-1}$. The idea is that we wish to show that $G + B$ is a bijection, and hence invertible. To do so set

$$E = G^{-1}B \Longrightarrow G + B = G^{-1}(\mathrm{Id} + G^{-1}B). \tag{16.2}$$

This is injective if $\mathrm{Id} + G^{-1}B$ is injective, and surjective if $\mathrm{Id} + G^{-1}B$ is surjective, since $G^{-1}$ is a bijection. From last time we know that

$$\|G^{-1}B\| < 1 \Longrightarrow \mathrm{Id} + G^{-1}B \text{ is invertible.} \tag{16.3}$$

Since $\|G^{-1}B\| \le \|G^{-1}\|\|B\|$ this follows if $\|B\| < \|G^{-1}\|^{-1}$ as anticipated. □

Thus $\mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, the set of invertible elements, is open. It is also a group – since the inverse of $G_1G_2$ if $G_1$, $G_2 \in \mathrm{GL}(\mathcal{H})$ is $G_2^{-1}G_1^{-1}$.

This group of invertible elements has a smaller subgroup, $\mathrm{U}(\mathcal{H})$, the unitary group, defined by

$$\mathrm{U}(\mathcal{H}) = \{U \in \mathrm{GL}(\mathcal{H}); U^{-1} = U^*\}. \tag{16.4}$$

The unitary group consists of the linear isometric isomorphisms of $\mathcal{H}$ onto itself – thus

$$(Uu, Uv) = (u, v),\ \|Uu\| = \|u\| \ \forall\ u, v \in \mathcal{H},\ U \in \mathrm{U}(\mathcal{H}). \tag{16.5}$$

This is an important object and we will use it a little bit later on.

The unitary group on a separable Hilbert space may seem very similar to the familiar unitary group of $n \times n$ matrices, $\mathrm{U}(n)$. It is, of course it is much bigger for one thing. In fact there are some other important differences which I will describe a little later on (or get you to do some of it in the problems). On important fact that you should know, even though I will not prove it, is that $\mathrm{U}(\mathcal{H})$ is contractible as a metric space – it has no significant topology. This is to be constrasted with the $\mathrm{U}(n)$ which have a lot of topology, and not at all simple spaces – especially for large $n$. One upshot of this is that $\mathrm{U}(\mathcal{H})$ does not look much like the limit of the $\mathrm{U}(n)$ as $n \to \infty$.

Now, for the rest of today I will talk about the opposite of the 'big' operators such as the elements of $\mathrm{GL}(\mathcal{H})$.

*Definition* 7. An operator $T \in \mathcal{B}(\mathcal{H})$ is of *finite rank* if its range has finite dimension (and that dimension is called the rank of $T$); the set of finite rank operators is denoted $\mathcal{R}(\mathcal{B})$.

Why not $\mathcal{F}(\mathcal{B})$? Because we want to use this for the *Fredholm operators*.

Clearly the sum of two operators of finite rank has finite rank, since the range is contained in the sum of the ranges (but is often smaller):

$$(T_1 + T_2u) \in \operatorname{Ran}(T_1) + \operatorname{Ran}(T_2). \tag{16.6}$$

Since the range of a constant multiple of $T$ is contained in the range of $T$ it follows that the finite rank operators form a linear subspace of $\mathcal{B}(\mathcal{H})$.

It is also clear that

$$B \in \mathcal{B}(\mathcal{H}) \text{ and } T \in \mathcal{R}(\mathcal{B}) \text{ then } BT \in \mathcal{R}(\mathcal{B}). \tag{16.7}$$

Indeed, the range of $BT$ is the range of $B$ restricted to the range of $T$ and this is certainly finite dimensional since it is spanned by the image of a basis of $\operatorname{Ran}(T)$. Similalry $TB \in \mathcal{R}(\mathcal{H})$ since the range of $TB$ is contained in the range of $T$. Thus we have in fact proved most of

**Proposition 24.** *The finite rank operators form a $*$-closed ideal in $\mathcal{B}(\mathcal{H})$, which is to say a linear subspace such that*

$$B_1,\ B_2 \in \mathcal{B}(\mathcal{H}),\ T \in \mathcal{R}(\mathcal{H}) \Longrightarrow B_1TB_2,\ T^* \in \mathcal{R}(\mathcal{H}). \tag{16.8}$$

*Proof.* In fact it is only the fact that $T^*$ is of finite rank if $T$ is which remains to be checked. To do this let us find an explicit representation for an operator of finite rank. First, since $\operatorname{Ran}(T)$ is finite dimensional, we can choose a basis, $f_i$ $i = 1, \dots, N$, for it. Then for any element $u \in \mathcal{H}$,

$$Tu = \sum_{i=1}^{N} c_i f_i. \tag{16.9}$$

The constants $c_i$ are determined, since the $f_i$ are a basis, and so define linear functionals $u \longmapsto c_i$. These are continuous. In fact we can simply choose the $f_i$ to be orthonormal and then, pairing (16.9) with $f_j$ we see that

$$c_j = (Tu, f_j) = (u, T^* f_j). \tag{16.10}$$

In particular there are elements (really by Riesz' theorem) $e_i = T^* f_i \in \mathcal{H}$ sucht that

$$Tu = \sum_{i=1}^{N} (u, e_i) f_i. \tag{16.11}$$

Conversely, if $T$ can be written in the form (16.11) then it is of finite rank, since its range is contained in the span of the $f_i$.

From (16.11) it follows that $T^*$ is also of finite rank since

$$(T^*v, u) = (v, Tu) = \sum_{j=1}^{N} (v, f_i)(e_i, u) \ \forall\ u \in \mathcal{H} \Longrightarrow T^*v = \sum_{i=1}^{N} (u, f_i) e_i. \tag{16.12}$$

The rôles of the $f_i$ and $e_i$ are simply interchanged. □

Next time I will show that the closure of the ideal $\mathcal{R}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ is the ideal of compact operators. Of course this closure is certainly closed(!) Moreover it is a $*$-closed ideal, since $T_n \to K$ in norm and $B_1$, $B_2 \in \mathcal{B}(\mathcal{H})$ implies

$$B_1 T_n B_2 \to B_1 K B_2,\ T_n^* \to K^*. \tag{16.13}$$

So, once we prove that the compact operators are the closure of the finite rank operators we will know that they form a closed, $*$-ideal.

Notice that the importance of the ideal condition – it is the analogue of the normal condition for a subgroup – is that the quotient $\mathcal{B}/\mathcal{I}$ of the algebra by an ideal is again an algebra. The quotient by the ideal, $\mathcal{K}(\mathcal{H})$, of compact operators is a Banach space since $\mathcal{K}$ is closed. It is called the Calkin algebra.

**Lemma 11** (Row rank=Colum rank). *For any finite rank operator on a Hilbert space, the dimension of the range of $T$ is equal to the dimension of the ranfe of $T^*$.*

*Proof.* We showed that a finite rank operator $T$ always takes the form (16.11). If the $f_i$ are taken to be a basis for the range of $T$, so $N = \dim \operatorname{Ran}(T)$, then the $e_i$ must be linearly independent. Indeed, if not then one of the $e_i$ can be replaced by a linear combination $e_i = \sum\limits_{j \neq i} c_j e_j$. Inserting this into (16.11) shows that

$$Tu = \sum_{j \neq i} (u, e_j)(f_j + \overline{c_j} f_i) \tag{16.14}$$

from which it follows that the range has dimension at most $N-1$ – which contradicts the choice of the $f_i$.

Since the $e_i$ are independent it follows from (16.12) that the range of $T^*$ has dimension $N$ (since the $f_i$ are independent) – if you like just say $\dim \operatorname{Ran}(T^*) \leq N$ for all finite rank $T$ and then use the fact that $(T^*)^* = T$ to deduce equality. $\square$

## Problem set 8, Due 11AM Tuesday 14 April.

Okay, I forgot to put the problems up. So, here are three problems that should be reasonably quick. If anyone is seriously inconvenienced by the limited time they have to work on them, just let me know and I will give you a couple of days.

*Problem 8.1* Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{16.15}$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

*Problem 8.2*

Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{16.16}$$

Show that $T$ is bounded on $L^2$ (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as $K$ in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of $x$ with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of $x$ and $y$ given by the previous problem, by truncating the Fourier series (in $y$) at some point $n$. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as $n$ becomes large. It might actually be clearer to do this the other way round, exchanging the roles of $x$ and $y$.

*Problem 8.3* Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx + ily)/2\pi$, $k, l \in \mathbb{Z}$, form a complete orthonormal basis.

## Solutions to Problem set 7

*Problem 7.1* Question:- Is it possible to show the completeness of the Fourier basis

$$\exp(ikx)/\sqrt{2\pi}$$

by computation? Maybe, see what you think. These questions are also intended to get you to say things clearly.

(1) Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{16.17}$$

for each fixed $t \in (0, 2\pi)$.

(2) Explain why this Fourier series converges to $f_t$ in $L^2(0, 2\pi)$ if and only if

$$2 \sum_{k>0} |c_k(t)|^2 = 2\pi t - t^2, \ t \in (0, 2\pi). \tag{16.18}$$

(3) Write this condition out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of $k^{-2}$ and $k^{-4}$.

(4) Can you explain how reversing the argument, that knowledge of the sums of these two series should imply the completeness of the Fourier basis? There is a serious subtlety in this argument, and you get full marks for spotting it, without going ahead a using it to prove completeness.

*Problem 7.2* Prove that for appropriate constants $d_k$, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$.

Hint: The usual method is to use the basic result from class plus translation and rescaling to show that $d'_k \exp(ikx/2)$ $k \in \mathbb{Z}$ form an orthonormal basis of $L^2(-2\pi, 2\pi)$. Then extend functions as odd from $(0, 2\pi)$ to $(-2\pi, 2\pi)$.

*Problem 7.3* Let $e_k$, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, $H$. Show that there is a uniquely defined bounded linear operator $S : H \longrightarrow H$, satisfying

$$Se_j = e_{j+1} \ \forall \ j \in \mathbb{N}. \tag{16.19}$$

Show that if $B : H \longrightarrow H$ is a bounded linear operator then $S + \epsilon B$ is *not* invertible if $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$.

Hint:- Consider the linear functional $L : H \longrightarrow \mathbb{C}$, $Lu = (Bu, e_1)$. Show that $B'u = Bu - (Lu)e_1$ is a bounded linear operator from $H$ to the Hilbert space $H_1 = \{u \in H; (u, e_1) = 0\}$. Conclude that $S + \epsilon B'$ is invertible as a linear map from $H$ to $H_1$ for small $\epsilon$. Use this to argue that $S + \epsilon B$ cannot be an isomorphism from $H$ to $H$ by showing that either $e_1$ is not in the range or else there is a non-trivial element in the null space.

*Problem 7.4* Show that the product of bounded operators on a Hilbert space is strong continuous, in the sense that if $A_n$ and $B_n$ are strong convergent sequences of bounded operators on $H$ with limits $A$ and $B$ then the product $A_n B_n$ is strongly convergent with limit $AB$.

Hint: Be careful! Use the result in class which was deduced from the Uniform Boundedness Theorem.