18.102 Introduction to Functional Analysis

Spring 2009

Lecture 7. Thursday, Feb 26

So, what was it with my little melt-down? I went too cheap on the monotonicity theorem and so was under-powered for Fatou's Lemma. In my defense, I was trying to modify things on-the-fly to conform to how we are doing things here. I should also point out that at least one person in the audience made a comment which amounted to pointing out my error.

So, here is something closer to what I should have said – it is not far from what I did say of course.

Proposition 12. *[Montonicity again] If $f_j \in \mathcal{L}^1(\mathbb{R})$ is a monotone sequence, either $f_j(x) \geq f_{j+1}(x)$ for all $x \in \mathbb{R}$ and all j or $f_j(x) \leq f_{j+1}(x)$ for all $x \in \mathbb{R}$ and all j, and $\int f_j$ is bounded then*

$$\tag{7.1} \{x \in \mathbb{R};\ \lim_{j \to \infty} f_j(x)\ \text{is finite}\} = \mathbb{R} \setminus E$$

where E has measure zero and

$$\tag{7.2} f = \lim_{j \to \infty} f_j(x)\ \text{a.e. is an element of}\ \mathcal{L}^1(\mathbb{R})$$
$$\text{with}\ \lim_{j \to \infty} \int |f - f_j| = 0.$$

Moral of the story – drop the assumption of positivity and replace it with the bound on the integral. In the approach through measure theory this is not necessary because one has the concept of a measureable, non-negative, function for which the integral 'exists but is infinite' – we do not have this.

Proof. Since we can change the sign of the f_i (now) it suffices to assume that the f_i are monotonically increasing. The sequence of integrals is therefore also montonic increasing and, being bounded, converges. Thus we can pass to a subsequence $g_i = f_{n_i}$ with the property that

$$\tag{7.3} \int |g_j - g_{j-1}| = \int g_j - \int g_{j-1} < 2^{-j}\ \forall\ j > 1.$$

This means that the series $h_1 = g_1$, $h_j = g_j - g_{j-1}$, $j > 1$, is absolutely summable. So we know for the result last time that it converges a.e., that the limit, f, is integrable and that

$$\tag{7.4} \int f = \lim_{j \to \infty} \int \sum_{k=1}^{j} h_k = \lim_{j \to \infty} \int g_j = \lim_{n \to \infty} \int f_j.$$

In fact, everywhere that the series $\sum_j h_j(x)$, which is to say the sequence $g_k(x)$, converges so does $f_n(x)$, since the former is a subsequence of the latter which is monotonic. So we have (7.1) and the first part of (7.2). The second part, corresponding to convergence for the equivalence classes in $L^1(\mathbb{R})$ follows from monotonicity, since

$$\tag{7.5} \int |f - f_j| = \int f - \int f_j \to 0\ \text{as}\ j \to \infty.$$

$\square$

Now, to Fatou's Lemma. This really just takes the monotonicity result and applies it to a general sequence of integrable functions with bounded integral. You should recall – as I did – that the max and min of two integrable functions is integrable and that

$$(7.6) \qquad \int \min(f, g) \leq \min(\int f, \int g).$$

Lemma 3. *[Fatou]. Let $f_j \mathcal{L}^1(\mathbb{R})$ be a sequence of non-negative (so real-valued integrable) functions such that $\int f_j$ is bounded above in $\mathbb{R}$, then*

$$f(x) = \liminf_{n\to\infty} f_n(x) \ \text{exists a.e.,} \ f \in \mathcal{L}^1(\mathbb{R}) \ \text{and}$$

$$(7.7)$$

$$\int \liminf f_n \leq \liminf \int f_n.$$

Proof. You should remind yourself of the properties of liminf as necessary! Fix k and consider

$$(7.8) \qquad F_{k,n} = \min_{k \leq p \leq k+n} f_p(x) \in \mathcal{L}^1(\mathbb{R})$$

as discussed briefly above. Moreover, this is a decreasing sequence, as n increases, because the minimum is over an increasing set an all elements are non-negative. Thus the integrals are bounded below by 0 so the monotonicity result above applies and shows that

$$(7.9) \qquad g_k(x) = \inf_{p \geq k} f_p(x) \in \mathcal{L}^1(\mathbb{R}), \ \int g_k \leq \int f_n \ \forall \ n \geq k.$$

Note that for a decreasing sequence of non-negative numbers the limit exists everywhere and is indeed the infimum. Thus in fact,

$$(7.10) \qquad \int g_k \leq \liminf \int f_n.$$

Now, let k vary. Then, the infimum in (7.9) is over a set which decreases as k increases. Thus the $g_k(x)$ are increasing. The integral is always bounded by one of the $\int f_n$ and hence is bounded above independent of k since we assumed a bound on the $\int f_n$'s. So, now we can apply the monotonicity result again to see that

$$f(x) = \lim_{k\to\infty} g_k(x) \ \text{exists a.e and} \ f \in \mathcal{L}^1(\mathbb{R}) \ \text{has}$$

$$(7.11)$$

$$\int f \leq \liminf \int f_n.$$

Since $f(x) = \liminf f_n(x)$, by definition of the latter, we have proved the Lemma. □

Now, we apply Fatou's Lemma to prove what we are really after:-

Theorem 2. *[Lebesgue's dominated convergence]. Suppose $f_j \in \mathcal{L}^1(\mathbb{R})$ is a sequence of integrable functions such that*

$$\exists \ h \in \mathcal{L}^1(\mathbb{R}) \ \text{with} \ |f_j(x)| \leq h(x) \ \text{a.e. and}$$

$$(7.12)$$

$$f(x) = \lim_{n\to\infty} f_j(x) \ \text{exists a.e.}$$

Then $f \in \mathcal{L}^1(\mathbb{R})$ and $\int f = \lim_{n\to\infty} \int f_n$ (including the assertion that this limit exists).

Proof. First, we can assume that the f_j are real since the hypotheses hold for its real and imaginary parts and together give the desired result. Moroever, we can change all the f_j's to make them zero on the set on which the estimate in (7.12) does not hold. Then this bound on the f_j's becomes

$$(7.13) \qquad -h(x) \leq f_j(x) \leq h(x) \ \forall \ x \in \mathbb{R}.$$

In particular this means that $g_j = g - f_j$ is a non-negative sequence of integrable functions and the sequence of integrals is also bounded, since (7.12) also implies that $\int |f_j| \leq \int h$, so $\int g_j \leq 2 \int h$. Thus Fatou's Lemma applies to the g_j. Since we have *assumed* that the sequence $g_j(x)$ converges a.e. to f we know that

$$h - f(x) = \liminf g_j(x) \text{ a.e. and}$$

$$(7.14) \qquad \int h - \int f \leq \liminf \int (h - f_j) = \int h - \limsup \int f_j.$$

Notice the change on the right from liminf to limsup because of the sign.

Now we can apply the same argument to $g'_j(x) = h(x) + f_j(x)$ since this is also non-negative and has integrals bounded above. This converges a.e. to $h(x) + f(x)$ so this time we conclude that

$$(7.15) \qquad \int h + \int f \leq \liminf \int (h + f_j) = \int h + \liminf \int f_j.$$

In both inequalities (7.14) and (7.15) we can cancel and $\int h$ and combining them we find

$$(7.16) \qquad \limsup \int f_j \leq \int f \leq \liminf \int f_j.$$

In particular the limsup on the left is smaller than, or equal to, the liminf on the right, for the same real sequence. This however implies that the are equal and that the sequence $\int f_j$ converges (look up properties of liminf and limsup if necessary ...). Thus indeed

$$(7.17) \qquad \int f = \lim_{n \to \infty} \int f_n.$$

$\square$

Generally in applications it is Lebesgue's dominated convergence which is used to prove that some function is integrable.

Finally I want to make sure that we agree that $L^1(\mathbb{R})$ is a Banach space. Note once again that I have used the somewhat non-standard notation

$$(7.18) \qquad \mathcal{L}^1(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{C}; f \text{ is integrable.}\}$$

This is a curly 'L'. We know that $f \in \mathcal{L}^1(\mathbb{R})$ implies that $|f| \in \mathcal{L}^1(\mathbb{R})$ (if you are wondering the converse might not be true if f oscillates badly enough). Now, we know exactly when the integral of the absolute value vanishes. Namely

$$(7.19) \quad \mathcal{N} = \{f \in \mathcal{L}^1(\mathbb{R}); \int |f| = 0\}$$

$$= \{f : \mathbb{R} \longrightarrow \mathbb{C}; f(x) = 0 \ \forall \ x \in \mathbb{R} \setminus E, \ E \text{ of measure zero}\}.$$

Namely, this is the linear space of null functions. We then defined

$$(7.20) \qquad L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}.$$

This has a non-curly 'L' – the notation is by no means standard but the definition (7.20) certainly is. Thus the elements of $L^1(\mathbb{R})$ are *equivalence classes of functions*

$$(7.21) \qquad\qquad [f] = f + \mathcal{N}, \ f \in \mathcal{L}^1(\mathbb{R}).$$

That is, we 'identify' to element of $\mathcal{L}^1(\mathbb{R})$ if (and only if) there difference is null, which is to say they are equal off a set of measure zero. Note that the set is not fixed, but can depend on the functions. Anyway, for an element of $L^1(\mathbb{R})$ the integral of the absolute value is well-defined:-

$$(7.22) \qquad\qquad \|[f]\|_{L^1} = \int |f|$$

since the right side is independent of which representative we choose.

Theorem 3. *The function $\| \cdot \|_{L^1}$ in (7.22) is a norm on $L^1(\mathbb{R})$ with respect to which it is a Banach space.*

The integral of the absolute value, $\int |f|$ is a *semi-norm* on $\mathcal{L}^1(\mathbb{R})$ – it satisfies all the properties of a norm except that $\int |f| = 0$ does not imply $f = 0$, only $f \in \mathcal{N}$. We are 'killing' this problem by taking the quotient.

Proof. I will not go through the proofs of the norm properties but you should. So, the only issue remaining is the completeness of $L^1(\mathbb{R})$ with respect to $\|\cdot\}_{L^1}$.

The completeness is a direct consequence of the Theorem in the last lecture on absolutely summable series of Lebesgue functions, so remind yourself of what this says. Also recall how we showed that if f is integrable, so is $|f|$. Namely, if f_j is an absolutely summable series (originally of step functions, now of Lebesgue integrable functions) then we defined

$$(7.23) \qquad\qquad g_1 = |f_1|, \ g_j = |\sum_{k \le j} f_k| - |\sum_{k \le j-1} f_k|$$

and observed that

$$(7.24) \qquad\qquad |g_j| \le |f_j| \ \forall \ j.$$

Thus, g_j is also absolutely summable and everywhere $\sum_j f_j(x)$ converges,

$$(7.25) \qquad\qquad \sum_{j \le N}(x) = |\sum_{j \le N} f_j(x)| \to |f(x)| \text{ as } N \to \infty.$$

This shows that $|f| \in \mathcal{L}^1(\mathbb{R})$, but more than that since

$$(7.26) \qquad\qquad \int |f| = \lim_{N \to \infty} \int |\sum_{j \le N} f_j(x)| \le \sum_j \int |f_j|.$$

Roughly speaking this is why we have been using absolutely summable series from the beginning.

So, going back to f_j and absolutely summable series in $\mathcal{L}^1(\mathbb{R})$, in the sense that $\sum_j |f_j|$, we can apply the discussion above to the truncated series starting at point N. Namely, the f_j for $j \ge N$ give an absolutely convergent series which sums a.e. to

$$(7.27) \qquad\qquad f(x) - \sum_{j < N} f_j(x) = \sum_{j > N} f_j(x).$$

Now, applying (7.26) we see that

$$(7.28) \qquad \int |f(x) - \sum_{j<N} f_j(x)| \le \sum_{j>N} \int |f_j|.$$

However, the absolute convergence means that the tail on the right is small with N, that is,

$$(7.29) \qquad \lim_{N \to \infty} \int |f - \sum_{j<N} f_j| = 0.$$

So, finally it is only necessary to think about $L^1(\mathbb{R})$ instead of $\mathcal{L}^1(\mathbb{R})$. An absolutely summable sequence in F_j in $L^1(\mathbb{R})$ is a series of equivalence classes $f_j + \mathcal{N}$ where $f_j \in \mathcal{L}^1(\mathbb{R})$. The absolutely summability condition is

$$(7.30) \qquad \sum_j \|F_j\|_{L^1} = \sum_j \int |f_j| < \infty$$

is what we need to start the discussion above. Namely, we have shown that the sum a.e. f of the series f_j is an element of $\mathcal{L}^1(\mathbb{R})$ and (7.29) holds. But this just means that the equivalenct class $F = f + \mathcal{N}$ satisfies

$$(7.31) \qquad \lim_{N \to \infty} \|F - \sum_{j<N} F_j\|_{L^1} = \lim_{N \to \infty} \int |f - \sum_{j<N} f_j| = 0.$$

Thus, $\sum_{j=1}^{N} F_j = F$ in $L^1(\mathbb{R})$ which is therefore complete. $\qquad \square$

Note that despite the fact that it is technically incorrect, everyone says '$L^1(\mathbb{R})$ is the space of Lebesgue integrable functions' even though it is really the space of equivalence classes of these functions modulo equality almost everywhere. Not much harm can come from doing this.