Problem Set 1

1. Let m and n be positive integers with no common factor. Prove that if $\sqrt{m/n}$ is rational, then m and n are both perfect squares, that is to say there exist integers p and q such that $m = p^2$ and $n = q^2$. (This is proved in Proposition 9 of Book X of Euclid's *Elements*). (3 points)

2. Problem 8 from page 22. (2 points)

3. Problem 9 from page 22. (2 points)

4. Let $\mathbb{R}$ be the set of real numbers and suppose $f \colon \mathbb{R} \to \mathbb{R}$ is a function such that for all real numbers x and y the following two equations hold

$$f(x + y) = f(x) + f(y), \tag{1}$$

$$f(xy) = f(x)f(y). \tag{2}$$

Claim: $f(x) = 0$ for all x or $f(x) = x$ for all x.
Prove this claim using the following steps:

(a) Prove that $f(0) = 0$ and that $f(1) = 0$ or 1.

(b) Prove that $f(n) = nf(1)$ for every integer n and then that $f(n/m) = (n/m)f(1)$ for all integers n, m such that $m \neq 0$. Conclude that either $f(q) = 0$ for all rational numbers q or $f(q) = q$ for all rational numbers q.

(c) Prove that f is nondecreasing, that is to say that $f(x) \geq f(y)$ whenever $x \geq y$ for any real numbers x and y.

(d) Prove that if $f(1) = 0$ then $f(x) = 0$ for all real numbers x. Prove that if $f(1) = 1$ then $f(x) = x$ for all real numbers x.

When writing the answer for this problem, please pay particular attention to completeness of the argument; and to structure, clarity and legibility of writing. Typesetting (LaTeX or similar) is encouraged, but not (yet) required. Your answer will be assessed by the grader for correctness, and then again by the recitation instructor for quality of exposition. (5 points)

Total: 3+2+2+5 = 12 points.

MIT OpenCourseWare
http://ocw.mit.edu

18.100C Real Analysis
Fall 2012

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.