SOLUTIONS TO PS4

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Solution/Proof of Problem 1. *Consider the open set*

$$B_n = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 - \frac{1}{n}\right\}.$$

Then we can see that $E \subset \cup B_n$ because for any point $(x, y) \in E$, $x^2 + y^2 < 1$, we can find an n big enough such that $x^2 + y^2 < 1 - \frac{1}{n}$, i.e. $(x, y) \in B_n$.

It is easy to see there is no finite subcover.

Solution/Proof of Problem 2. *At first, from the definition, we have $d(x, x) = 0$. From the equality*

$$d(x, y) = ||x|| + ||y|| = ||y|| + ||x|| = d(y, x),$$

we have $d(x, y) = d(y, x)$. We also have

$$d(x, y) = ||x|| + ||y|| = (\sum_{i=1}^{n} x_i^2)^{1/2} + (\sum_{i=1}^{n} y_i^2)^{1/2} \geq 0$$

and it is easy to see that $d(x, y) = 0$ iff $x = y = 0$.

From

$$||x|| + ||y|| \leq ||x|| + ||z|| + ||z|| + ||y||$$

we have $d(x, y) \leq d(x, z) + d(y, z)$.

So d is a metric on $\mathbb{R}^n$.

Open set in $(\mathbb{R}^k, d)$ may be not open in $(\mathbb{R}^k, d_{Euclid})$. For example, consider the open ball in $(\mathbb{R}^k, d)$,

$$B_r(x) = \{y \in \mathbb{R}^k : d(x, y) < r\}.$$

When $r > ||x||$, then we have

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^k : d(x, y) < r\} = \{y \in \mathbb{R}^k : ||x||| + ||y|| < r\} \\ &= \{y \in \mathbb{R}^k : ||y|| < r - ||x||\}, \end{aligned}$$

is just the open ball $B_{r-||x||}(0)$ under the Euclidean metric.

But when $r < ||x||$, then we have

$$B_r(x) = \{y \in \mathbb{R}^k : d(x, y) < r\} = \{y \in \mathbb{R}^k : |x||| + ||y|| < r\} = \{x\},$$

is not open under the Euclidean metric.

Conversely, consider an open set $U \subset (\mathbb{R}^k, d_{Euclid})$. Since for any point $x \neq 0$, under the new metric, $B_r(x) = \{x\} \subset U$ for any $r < ||x||$; if $x = 0$, then $B_r(0) \subset \mathbb{R}^k$ for some small r under the new metric. So we can always find an open neighborhood of x in the new metric that is contained in U. This means U is open in the new metric.

Solution/Proof of Problem 3. *First, recall that in any metric space a finite set is compact. We will show that for the discrete metric, these are the only compact sets.*

Notice that every subset $\{x\}$ which contains only one point in X is an open subset. Indeed, if $Y = \{x\}$, then, for any $r < 1$, we have $B_r(x) = \{x\}$ hence $B_r(x) \subset Y$, i.e., Y is open.

Now suppose Y is a compact subset of X. We can consider an open cover $Y \subset \bigcup_{y \in Y} U_y$, where $U_y = \{y\}$. Since Y is compact, this cover has a finite subcover. So $Y \subset \bigcup_{\substack{\text{finitely many} \\ y \in Y}} U_y$. Hence Y is a finite set.

Solution/Proof of Problem 4. *From the definition of E, we can see that $E \subset \{x \in \mathbb{Q}, -3 < x < 3\}$. So it is bounded.*

E **is closed**. *Recall that, in any metric space, a set E is closed if and only if its complement is open. If x is any point whose square is less than 2 or greater than 3 then it is clear that there is a nieghborhood around x that does not intersect E. Indeed, take any such neighborhood in the real numbers and then intersect with the rational numbers. So the only problem would be at points whose square is exactly 2 or 3, but we know that are no such points within the rational numbers.*

E **is not compact**. *Consider the open cover*

$$U_n = \{x \in \mathbb{Q} : 2 + 1/n < x^2 < 3 - 1/n\}, n \in \mathbb{N}.$$

It is easy to see that it has no finite subcover.

E **is open**. *Given any point $x \in E$ there is a neighborhood of x within the real numbers of elements whose square is between 2 and 3, intersect this with the rational numbers to see that E is an open subset of $\mathbb{Q}$.*

Solution/Proof of Problem 5. *Suppose X and Y are two compact sets. If $\{U_\alpha\}$ is an open cover for $X \cup Y$, then it is also an open cover of X. Since X is compact, there is a finite subcover $\{U_\beta\}_{\beta \in I} \subset \{U_\alpha\}$ which still covers X. Similarly, we also have a finite subcover $\{U_\beta\}_{\beta \in J} \subset \{U_\alpha\}$ which covers Y. Putting these covers together,*

$$\{U_\beta\}_{\beta \in I \cup J} \subset \{U_\alpha\},$$

we get a finite subcover of $X \cup Y$. So by the definition, $X \cup Y$ is compact.

Since X is a compact set in a metric space, it is closed. Hence $X \cap Y$ is the intersection of a closed set with a compact set. From Theorem 2.35's corollary, we can see that $X \cap Y$ is compact.

Solution/Proof of Problem 6. *The set $\{1\}$ has no limit points because any neighborhood of this point has only one element 1.*

The statement can be proved as the follows. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space with infinitely many distinct elements. Suppose $\lim_{n \to \infty} x_n = x_0$. Then by the definition of the limits, for any neighborhood of x_0, there are at most finitely many points in the sequence outside the neighborhood. So we can always choose a point different from x_0 in any neighborhood which means x_0 is a limit point of the set.

Solution/Proof of Problem 7. *Because they are countable, it is possible to put the rational numbers in $[0,1]$ in a sequence, (p_n). We claim that every point $x \in [0,1]$ is a limit of a subsequence of (p_n).*

We proved in class that between any two real numbers there is a rational number, it follows that between any two real numbers there are infinitely many rational numbers. This allows us to construct a subsequence of p_n converging to x as follows. Assume for simplicity that x and $x+\frac{1}{10}$ are both in $[0,1]$. From among the infinitely many rational numbers between x and $x+\frac{1}{10}$, let p_{n_1} be the first of the p_n to fall in $x < p_{n_1} \leq x + \frac{1}{10}$. Because there are inifinitely many rational numbers between x and p_{n_1} we can pick p_{n_2} to be the first rational number in the sequence p_n ocurring after p_{n_1} to fall inside $x < p_{n_2} \leq x + \frac{1}{10^2}$. Similarly, we choose p_{n_3} to be the first rational number in the sequence p_n ocurring after p_{n_2} to fall inside $x < p_{n_3} \leq x + \frac{1}{10^3}$. Continuing in this fashion, we achieve a subsequence of p_n, which we denote p_{n_k} with the property that

$$x < p_{n_k} \leq x + \frac{1}{10^k}, \text{ for any } k \in \mathbb{N}.$$

Hence $p_{n_k} \to x$.

This was done under the assumption that both x and $x+\frac{1}{10}$ were both in $[0,1]$. If that is not the case, but $x < 1$ then we can find N such that x and $x+\frac{1}{10^N}$ are both in $[0,1]$ and we can start the construction from there. Finally, if $x = 1$ then x and $x - \frac{1}{10}$ are both in $[0,1]$ and we can carry out the above construction requiring that p_{n_k} satisfy $x - \frac{1}{10^k} \leq p_{n_k} < x$ for every k.

Hence in every case we obtain a subsequence of (p_n) that converges to x.

Solution/Proof of Problem 8. *The question is asking: If A is connected, does the interior of A have to be connected? does the closure of A have to be connected?*

Closure of a connected set is always connected. Suppose $\overline{E} = A \cup B$, where $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$, we show that $\overline{E}$ is connected by proving that either A or B must be empty.

We know that E is connected and $E = (A \cap E) \cup (B \cap E)$ with $A \cap E$, $B \cap E$ separated sets, hence we must have $A \cap E = \emptyset$ or $B \cap E = \emptyset$. Say that $A \cap E = \emptyset$, then $E \subseteq B$ and hence $\overline{E} \subseteq \overline{B}$. But we know that $A \cap \overline{B} = \emptyset$, hence

$$A = A \cap (A \cup B) = A \cap \overline{E} \subseteq A \cap \overline{B} = \emptyset,$$

which implies that $\overline{E}$ is connected.

The interior of a connected set may not be connected. Consider two tangent closed disk in $\mathbb{R}^2$: $\overline{B_1((0,1))}$ and $\overline{B_1((0,-1))}$. The union will give us a connected set. But the interior part of it will be two separated open balls.

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