CONTINUOUS ALMOST EVERYWHERE

Definition 1. *Let Δ be a subset of $\mathbb{R}$. We say that Δ* has measure 0 *if, for each $\epsilon > 0$, there is a sequence of balls $(B_j = B_{r_j}(c_j)))_{j\in\mathbb{N}}$ with radii $r_j > 0$ (and centres $c_j \in \mathbb{R}$), such that $\Delta \subset \bigcup_{j\in\mathbb{N}} B_j$ and $\sum_{j=1}^{\infty} r_j < \epsilon$.*

The balls here are open intervals of lenght $2r_j$, so the series $\sum_{j=1}^{\infty} 2r_j < 2\epsilon$ is the sum of lengths. However, the balls could intersect each other, so 2ϵ is just an upper bound for the "total length" or "volume" (what we will call "measure") of their union $\bigcup_j B_j$. A full definition of "measures" is beyond the scope of this course; the only important property that we will use here is that subsets have smaller measure than the set that they are contained in (assuming both are "measurable"). So the above Definition just says that the measure of Δ is smaller than 2ϵ for any $\epsilon > 0$. So – if the measure of Δ is to be a nonnegative real number – the measure of Δ will indeed be 0.

Examples of measure 0 subsets of $\mathbb{R}$ are all finite and countable subsets (see Lemma below) as well as the (uncountable) Cantor set.

Lemma 2. *Let $\Delta \subset \mathbb{R}$ be a countable subset, then Δ has measure 0.*

Proof. By assumption, we can enumerate $\Delta = \{c_j \mid j \in \mathbb{N}\}$. Now for any $\epsilon > 0$ choose the sequence of radii $r_j = 2^{-j-1}\epsilon$, then clearly $\bigcup_{j\in\mathbb{N}} B_{r_j}(c_j)$ contains Δ (since it contains every $c_j \in \Delta$ as centre of a ball) and we have $\sum_{j=1}^{\infty} r_j = \sum_{k=0}^{\infty} 2^{-k}\frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon$. ∎

Given a function $f\colon [a,b] \to \mathbb{R}$, we say it is *continuous almost everywhere* if the set $\Delta_f \subset [a,b]$ of discontinuities of f has measure 0. If the set of discontinuities is finite or countable, then f is continuous almost everywhere, for example. So the function $f(x) = 1/x$ for $x \neq 0$ and 0 for $x = 0$ is continuous almost everywhere. However, this function is not Riemann integrable (on $[-1,1]$ say). Indeed, Riemann integrable functions must be bounded. But this is the only restriction, it turns out.

Theorem 3. *Let $-\infty < a < b < \infty$, and let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable iff it is continuous almost everywhere.*

Here, we will only prove the *reverse* direction:

Proposition 4. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded and continuous almost everywhere. Then f is Riemann integrable.*

The converse is no harder (in fact, in some ways it is easier), but it requires a somewhat different approach to the integral than Rudin takes.

In order to prove that f is Riemann integrable we need to find partitions $(x_i)_{i=0,\dots,N}$ such that the difference of upper and lower approximation becomes small:

$$U(f,(x_i)) - L(f,(x_i)) = \sum_{i=1}^{N} (x_i - x_{i-1}) \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right). \tag{1}$$

This is a sum over the oscillations of the function (i.e. the maximal difference of function values on each interval $[x_{i-1}, x_i]$), weighted with the length of the intervals. We will use oscillations throughout the proof, so let us introduce that notation. For any subset $B \subset \mathbb{R}$ we write

$$\mathrm{osc}_f(B) = \sup_{x \in B} f(x) - \inf_{x \in B} f(x).$$

With this notation we simply have $U(f,(x_i)) - L(f,(x_i)) = \sum_{i=1}^{N}(x_i - x_{i-1})\mathrm{osc}_f([x_{i-1}, x_i])$. Now the basic idea is to show that continuity almost everywhere means that large oscillations of the function only happen on very small sets, and then control those appropriately, so that the sum in (1) becomes small.

To embark on a precise proof of the Proposition, we fix $\epsilon > 0$, and let $B_j = B_{r_j}(c_j)$ be balls as in the definition of measure 0, which cover Δ_f and have small measure $2\sum_{j=1}^{\infty} r_j < 2\epsilon$. We then pick out those balls on which f has large oscillation: Let

$$J := \{ j \in \mathbb{N} \,|\, \mathrm{osc}_f(B_j) > \epsilon \} \qquad \text{and} \qquad V_\epsilon := \bigcup_{j \in J} B_j$$

be the index set and the union of just those balls. Note that the total length of V_ϵ cannot be larger than that of all balls B_j, that is (since all these sequences of nonnegative terms converge) $2\sum_{j\in J} r_j \le 2\sum_{j\in\mathbb{N}} r_j < 2\epsilon$.

Now we will try to find a partition such that each interval has either small oscillation or is contained in V_ϵ (which has small total length). We restrict ourselves to equidistant partitions $(x_i = a + (b-a)\frac{i}{N})_{i=0,\dots N}$ into intervals $[x_{i-1}, x_i]$ of length $\frac{b-a}{N}$. The key to the proof is the next Lemma, which says exactly that a sufficiently fine partition (with large $N \in \mathbb{N}$) satisfies our requirements (every interval has small oscillation or is entirely contained in V_ϵ).

Lemma 5. *There exists $N \in \mathbb{N}$ so that, for any $i \in \{1, \dots, N\}$, if $\mathrm{osc}_f([x_{i-1}, x_i]) > \epsilon$ then $[x_{i-1}, x_i] \subset V_\epsilon$.*

Proof. We argue by contradiction. If this Lemma is false, then for every $N \in \mathbb{N}$ we can find $i \in \{1, \dots, N\}$ such that $\mathrm{osc}_f([x_{i-1}, x_i]) > \epsilon$ but $[x_{i-1}, x_i] \cap V_\epsilon^c \neq \varnothing$. Hence, for every $N \in \mathbb{N}$ we find $s_N, t_N, z_N \in [x_{i-1}, x_i]$ (for some i) such that $\mathrm{osc}_f([x_{i-1}, x_i]) \ge f(s_N) - f(t_N) > \epsilon$ and $z_N \in V_\epsilon^c$. The sequence $(s_N)_{N\in\mathbb{N}}$ is bounded (it lies in $[a, b]$), so it has a convergent subsequence $\lim_{k\to\infty} s_{N_k} = y \in [a, b]$. Since the points t_N and z_N have distance at most $\frac{b-a}{N}$ from s_N, they all converge (for the same subsequence $(N_k)_{k\in\mathbb{N}}$) to the same limit y. The function f is discontinuous at the limit y since $f(s_{N_k}) - f(t_{N_k}) > \epsilon$ does not converge to 0. So we have $y \in \Delta_f$, and hence $y \in B_j$ for some $j \in \mathbb{N}$. On the other hand, $(z_{N_k})_{k\in\mathbb{N}}$ is a sequence in V_ϵ^c. Here V_ϵ is a union of open balls, hence open, and so its complement V_ϵ^c is closed, hence contains the limit $y = \lim_{k\to\infty} z_{N_k}$. So $y \notin V_\epsilon$ cannot be contained in any ball B_j of large oscillation, and hence $y \in B_j$ for some ball with $\mathrm{osc}_f(B_j) \le \epsilon$. However, the ball B_j is open and thus will contain s_{N_k} and t_{N_k} for all sufficiently large $k \in \mathbb{N}$. This leads to the contradiction

$$\epsilon < f(s_{N_k}) - f(t_{N_k}) \le \mathrm{osc}_f(B_j) \le \epsilon.$$

And this – magically – proves the Lemma. $\square$

Proof of Proposition: To prove that f is integrable, let any $\epsilon' > 0$ be given, then we need to find a partition P with $U(f,P) - L(f,P) < \epsilon'$. We will do this by following our above constuction, starting from a covering $\bigcup_{j\in\mathbb{N}} B_j$ of Δ_f with total length $< 2\epsilon$ for some $\epsilon > 0$ that we are free to choose. It will turn out that $\epsilon = \epsilon'((b-a)+2\mathrm{osc}_f([a,b]))^{-1}$ is a wise choice, where $\mathrm{osc}_f([a,b]) = \sup f - \inf f$, the total oscillation of f, is finite since f is bounded.

Now Lemma 5 provides an $N \in \mathbb{N}$ such that the partition $P = (x_i = a+(b-a)\frac{i}{N})_{i=0,\dots,N}$ has the following crucial property: Every interval $[x_{i-1}, x_i]$ satisfies at least one of the properties $\mathrm{osc}_f([x_{i-1}, x_i]) \leq \epsilon$ or $[x_{i-1}, x_i] \subset V_\epsilon$ (and any interval satisfies $\mathrm{osc}_f([x_{i-1}, x_i]) \leq \mathrm{osc}_f([a,b])$). With that we can estimate

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{N} \tfrac{b-a}{N} \,\mathrm{osc}_f([x_{i-1}, x_i]) \\
&\leq \sum_{i=1}^{N} \tfrac{b-a}{N}\,\epsilon \quad + \quad K\tfrac{b-a}{N}\,\mathrm{osc}_f([a,b]),
\end{aligned}
$$

where K is the number of $i \in \{1,\dots,N\}$ such that $[x_{i-1}, x_i] \subset V_\epsilon$. The total length ("measure") of all these intervals adds up to $K\frac{b-a}{N}$. On the other hand, these intervals have overlaps at most in some endpoints, and their union is contained in V_ϵ. Hence their measure should be bounded by that of V_ϵ. This is the point where we invoke measure theory to deduce $K\frac{b-a}{N} < 2\epsilon$. Once we have that, we easily obtain

$$
U(f,P) - L(f,P) < (b-a)\epsilon + 2\epsilon\,\mathrm{osc}_f([a,b]) = \epsilon'.
$$

That was exactly what we needed to show for any given $\epsilon' > 0$. So it follows by Theorem 6.6 in Rudin that f is Riemann integrable. $\square$

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