18.100A Assignment 1

Directions: This first assignment is diagnostic, to give some idea of how well you can write a coherent argument in decent style, find errors, use basic calculus and elementary mathematics, read a proof in the book and adapt it to an analogous question.

Therefore, **no collaboration, outside help (except from me), looking up problem sets from previous years, etc. is allowed on this first assignment.**

Reading:

Chapter 1: 1.1-1.3 and 1.6 go quickly and shouldn't be hard – 1.1 is for background motivation, and you can skip the argument for Theorem 1.3.

Sections 1.4 and 1.5 are more substantial; 1.5 is needed for Problem 2 below.

2.1-.3 are elementary, but also the most common source of error all semester.

Appendix A.1-A.4 Background, read as needed. Specific topics there (*induction*, *counterexample*, and *contraposition* are used in the Problems below.

This first problem uses induction (A.4), 1.1-1.3, and 1.6, and 2.1. It has several parts; later parts depend on the earlier ones. If you are uncertain about how to do one of the parts, you can still use what it states in the subsequent parts.

Problem 1. (3: .5,.5,1,.5,.5) Define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \sqrt{2x_n - 1}, \quad x_0 = a, \text{ where } a > 1 .$$

(Note: by convention, $\sqrt{b}$, for $b > 0$, always means the positive square root.)

a) Work Question 3(i) at the end of section 2.1 . (Follow the hint given.)
b) Prove by induction (App. A.4) that $x_n > 1$ for all $n \geq 0$.
c) Prove that $\{x_n\}$ is strictly decreasing.
d) Prove the sequence has a limit, using the ideas in Chapter 1.
e) Prove the limit is 1, for any $a > 1$.

(For part (e), you may assume that if b_n is a positive monotone sequence having the limit L, then $\sqrt{b_n}$ has the limit $\sqrt{L}$.)

In addition, you can use this elementary theorem about limits of monotone sequences:

Theorem. *If c is a constant, and $a_n, b_n, a_nb_n, a_n+b_n$ are bounded monotone sequences,*

$$\lim_{n\to\infty} c\,a_n = c \lim_{n\to\infty} a_n;$$
$$\lim_{n\to\infty} a_nb_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n;$$
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(If you used backwards reasoning (cf. Q1.4/1 Answer) in (b) or (c), any non-obvious steps of the form: [line k $\Rightarrow$ line (k-1)] should be justified.)

Problem 2. (3.5: 1.5,2) Read 1.5, and by adapting the arguments in the two proofs on page 9, using comparison with the area under a curve and calculus, prove the following:

Consider the strictly increasing sequence $x_n = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2}, \quad n \geq 1.$

a) Prove it is bounded above by 2; deduce that it has a limit $L \leq 2$.

b) Prove that the "tail" sequence $\{x_n\}$, $n \geq N$, is bounded below by $3/2$, for some number N.

(Since the sequence is increasing, this can be done by simply calculating x_n explicitly, for increasing values of n until it grows larger than $3/2$, but that won't teach you anything. Instead, use geometric reasoning and calculus, as in the proof of Prop. 1.5B.

Place the rectangles correctly on the picture so you can show the sum $\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$ is larger than the area under the curve from 2 to $n+1$, plus some triangles whose total area you can calculate, and in this way determine a value of N which works. (Don't forget the first rectangle, which this calculation omits.) What's the smallest such value of N?)

This shows that $3/2 \leq L \leq 2$. The actual value of L is $\pi^2/6 \approx 10/6$.

Problem 3. (3.5: 1,1,.5,1) The proof of Prop.1.4 begins with the inequality ($\geq$ includes $h = 0$):

$(1+h)^2 \geq 1 + 2h$, for all h. A generalization of this would be

$$(1+h)^n \geq 1 + nh, \quad \text{for all } h, \text{ and } n \geq 1. \tag{1}$$

Proof by induction: (cf. App. A.4, Examples A and B) The basis step $P(1)$ is trivial; the induction step $P(n) \Rightarrow P(n+1)$ is proved in three steps:

$$\begin{aligned}(1+h)^{n+1} &= (1+h)^n(1+h), \\ &\geq (1+nh)(1+h), \quad \text{since } P(n) \text{ is true}, \\ &\geq 1 + (n+1)h, \quad \text{since } nh^2 \geq 0.\end{aligned}$$

a) Find a counterexample to (1) (cf. App. A.3 for "counterexample").

b) Since the counterexample shows that (1) is false as stated, find and describe the error in the above proof.

c) Give the weakest hypothesis (i.e., condition) on h for which the proof above will be valid (cf. A.1 p. 405 for "weakest").

d) Find a value of h which does not satisfy the condition in (c), yet for which the inequality (1) is still valid for $n \geq 1$.

Remarks:

Part (b): if you're stuck for more than 10 minutes, take the *trace* of your counterexample in the proof – i.e., substitute the numerical values of n and h used by the counterexample into each successive line of the proof, and see where it goes wrong; then explain in general where the error lies.

Part (d): this shows that **the failure of a method of proof** – induction, in this case – **doesn't show that a statement is false**, since someone might come along with a different method of proof which succeeds.

Remember this; it will haunt us throughout the semester!