Converse of Sarkovskii's Theorem

Wes McKinney

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The Period Three Theorem and its remarkable generalization, Sarkovskii's theorem, provide a great deal of information about the periodic behavior of the iterates of continuous functions. First, we recall the Sarkovskii ordering of the positive integers.

$$\begin{aligned}
&3, 5, 7, 9, \cdots \\
&2 \cdot 3, 2 \cdot 5, 2 \cdot 7, \cdots \\
&\vdots \\
&2^m \cdot 3, 2^m \cdot 5, 2^m \cdot 7, \cdots \\
&\vdots \\
&\cdots, 2^3, 2^2, 2, 1.
\end{aligned}$$

We write $x \rhd y$ if x precedes y in this ordering. This ordering allows us to state Sarkovskii's Theorem.

Theorem 1 (Sarkovskii's Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose f has a periodic point of prime period k. If $k \rhd l$ in the Sarkovskii ordering, then f also has a periodic point of period l.*

Proof: The majority of this theorem is proved in Devaney's text, namely the cases k odd and $k = 2^m$. He leaves the case of $k = p \cdot 2^m$, for p odd, to the reader, so we will prove this here. Suppose $k = p \cdot 2^m$, then consider f^{2^m} (2^mth iterate of f). Then by the odd case of Sarkovskii's theorem, f^{2^m} has points with prime period $n \cdot 2^r$ (n odd), where either $r \geq 1$, or n is an odd integer greater than p (r could possibly be 0 if $n > p$). As a result, f has points with prime period $n \cdot 2^{m+r}$ for these same n and r. Along these lines, f will also have a point of prime period 2^{m+1}, and by the 2^l case of Sarkovskii's theorem, f will have points of prime period 2^k for all $k < m + 1$. $\square$

Our goal is to show that the "converse" of this theorem is actually true, that is:

Theorem 2 (Sarkovskii Converse). *For each $k \in \mathbb{Z}^+$, there exists $f : \mathbb{R} \to \mathbb{R}$ continuous such that for all $l \rhd k$ in the Sarkovskii ordering, f has a periodic point of prime period k but none of period l.*

We will be able to prove this converse once we exhibit four explicit constructions encompassing the possible numbers in the Sarkovskii ordering. We will present these constructions in a series of four lemmas, after which the theorem will follow.

Lemma 1. *For each $n \in \mathbb{Z}^+$, there exists $f : I \to I$ continuous from a closed interval to itself such that f has a point of prime period $2n+1$ but none of period $2n-1$.*

Proof: Define the following function:

$$f = \begin{cases} nx+1 & x \in [1,2] \\ -x+2n+3 & x \in [2,n+1] \\ -2x+3n+4 & x \in [n+1,n+2] \\ -x+2n+2 & x \in [n+2,2n+1] \end{cases}$$

First, we note the orbit of 1 under f is $n+1, n+2, n, n+3, n-1, n+4, n-2, n+5, \ldots, 2, 2n+1, 1$, so that 1 is a periodic point of prime period $2n+1$. Now we show that $f^{2n-1}([j,j+1]) \cap (j,j+1) = \emptyset$ except for $[n+1,n+2]$. Under repeated mapping of f for example:

$$[1,2] \to [n+1,2n+1] \to [1,n+2] \to [n,2n+1]$$
$$\to [1,n+3] \to [n-1,2n+1] \to \cdots \to [1,2n] \to [2,2n+1].$$

Similarly, $[2,3] \to [2n,2n+1] \to [1,2] \to \cdots \to [3,2n+1]$. And in general $[j,j+1] \to \cdots \to [j+1,2n+1]$. Hence none of these intervals have periodic points of period $2n-1$. When we map $[n+1,n+2]$, we get $[n,n+2] \to [n,n+3] \to [n-1,n+3] \to \cdots \to [1,2n+1]$, so that $f^{2n-1}([n+1,n+2])$ has a fixed point in the interval. But we have that f^{2n-1} is monotonically decreasing on that interval, since f is only increasing on $[1,2]$, hence the fixed point of f^{2n-1} is also the fixed point of f, which means that it is not a periodic point of prime period $2n-1$. It follows that f does not have any such points of period $2n-1$. $\square$

Lemma 2. *For each $k \in \mathbb{Z}^+$, there exists $f : I \to I$ continuous such that f has a point of prime period $2^k(2n+1)$ but none of period $2^k(2n-1)$.*

Proof: In order to exhibit such a function, we first devise a way to "double" the period of a point in a map. We do so by dividing tripling the size of the interval on which f is defined, then squeezing the graph of f on I into the upper-left corner of $3I \times 3I$. We then extend f piecewise linearly on the rest of the interval $3I$ in the following way. Thus, we define Double(f) in the following way:

$$F(x) = \begin{cases} f(x)+2h & x \in [0,h] \\ x-2h & x \in [h,3h] \end{cases}$$

and F is linear between h and $3h$, with f being originally defined on $[0,h]$. Now, if $x \in [0,h]$, then $F(x) = f(x)+2h \in [h,3h]$, so that $F^2(x) = f(x)$. We note that $F([0,h]) \subset [h,3h]$, and $F([h,3h]) = [0,h]$, so that points in either of those intervals are mapped back and forth between those intervals. By the first property of F that we proved, if $x \in [0,h]$ is a periodic point of prime period n of f, then since $F^2(x) = f(x)$, it is evident that this x will have period $2n$ under F. Conversely, suppose F has a point $q \in [0,h]$ of prime period $2n$. We note that $F^2(q) = q'$ implies $f(q) = q'$ by the above. Thus q has prime period n for f. For example if f originally had a point of prime period 5 but not 3, this guarantees that F will have a point of period 10 and not 6 (otherwise f would have a point of period 3, which it does not). Finally, since F is monotone decreasing on $[h, 3h]$, there are no periodic points in this interval.

To complete the proof, we double f from the previous lemma k times to get a function with a periodic point of of period $2^k(2n+1)$, but not of $2^k(2n-1)$. □

Lemma 3. *For each $n \in \mathbb{Z}^+$, there exists $f : I \to I$ continuous such that f has a point of period 2^n but not of 2^{n+1}.*

Proof: We start by constructing a function that has a point of period 1 but not period 2. Let $f(x) = x$, with $x \in [0, 1]$. Then x is monotone increasing and hence has no points of period greater than 1. Now by repeated application of the doubling construction from the previous lemma, we obtain a function with period 2^n but not 2^{n+1}. □

Lemma 4. *For each $n \in \mathbb{Z}^+$, there exists $f : I \to I$ continuous such that f has a point of period $3 \cdot 2^n$ but none of $(2m-1)2^{n-1}$, for all $m \in \mathbb{Z}^+$.*

Proof: We define a function $f : [1, 3] \to [1, 3]$ with $f(1) = 2, f(2) = 3, f(3) = 1$, with linearity between these points. Now f has a 3 cycle, so now if we double f, we obtain a point of period 6, and we must show that this F does not have any points of odd period. This follows very easily since $F([1, 3]) = [5, 7]$, $F([5, 7]) = [1, 3]$, and F is monotone decreasing on $[3, 5]$, hence the orbit of a point returns to $[1, 3]$ every other iterate, so that the period of any periodic point must be even.

Now we use induction and the result of Lemma 2, supposing that we have already proved this for $n-1$. Suppose we have doubled f n times, so that this new F_n has a periodic point of period $2^n \cdot 3$. Now suppose that F_n had a periodic point of period $2^{n-1}(2m-1)$ for some positive $m \in \mathbb{Z}$. Then F_{n-1} (the result of doubling f $n-1$ times) must have a periodic point of period $2^{n-2}(2m-1)$, which contradicts our inductive hypothesis. Thus, F does not have any points of period $2^{n-1}(2m-1)$ for any m. □

Now having proved these lemmas we can prove the Sarkovskii Converse. We note that the cases we have identified encompass the possible k's in the Sarkovskii ordering. These lemmas are sufficient since if any of the constructed functions had a periodic point two steps before k (where k is one of the numbers in the lemmas), then by Sarkovskii's Theorem, that function would have a periodic point of period l, where l immediately precedes k, which we know not to be true by construction. Thus, the converse is proved. ■

References

Devaney, Robert. *An Introduction to Chaotic Dynamical Systems.* Benjamin/Cummings, 1996.