

18.091 Lecture 9 Part 2

Chaos in Newton's Method, and Fractals

Alejandro Ochoa

April 27, 2005

1 Chaos in Newton's Method

Newton's Method is an iteration algorithm that allows us to numerically find roots of some functions. We have discussed some cases in which convergence to a root will not occur for certain selection of "initial guesses". It may also happen that convergence does not occur for any initial guess, as is the case of a polynomial with no roots.

Consider $F(x) = x^2 + 1$. The corresponding N is $N(x) = \frac{1}{2}(x - \frac{1}{x})$. We will show that this N is chaotic. Let D be the doubling function, and $C(x) = \cot(\pi x)$. Then

$$\begin{aligned} C(D(x)) &= \cot(\pi D(x)) = \cot(2\pi x) \\ &= \frac{\cos^2(\pi x) - \sin^2(\pi x)}{2\sin(\pi x)\cos(\pi x)} \\ &= \frac{1}{2}\left(\cot(\pi x) - \frac{1}{\cot(\pi x)}\right) \\ &= N(C(x)). \end{aligned}$$

Hence N is conjugate to D , which we know is chaotic. It follows that N is also chaotic.

2 Newton's Method: Speed of Convergence

It is noted in the book that for some functions F , its corresponding Newton iterating function N converges faster to a root of F than for other choices of functions F . A useful definition is that of a *superattracting* fixed point x_0 of N , which is a point with $N'(x_0) = 0$. Seeds converge faster to superattracting fixed points than to regular fixed points under iteration.

3 Fractals: The Chaos Game

Fractals, as we will see, are static subsets of euclidean space with very interesting properties. Perhaps the most interesting is that fractals can arise naturally in chaotic systems. Consider the cartesian plane. Choose an equilateral triangle with vertices A, B, and C, and pick another point p at random. The game consists on generating a sequence of points p_i recursively as follows:

- $p_0 = p$.
- Choose a random vertex of the triangle ABC, then p_{i+1} is the midpoint of the segment connecting p_i and the chosen vertex.

At first it seems that the resulting collection of points $\{p_i\}$ depends heavily on the choices one makes for the vertex at every step in the recursion. However, that collection is contained (with probability 1) in a fractal called the Sierpinski triangle.

4 Examples of Fractals

Definition. A *fractal* is a subset of \mathbf{R}^n which is self-similar and whose fractal dimension exceeds its topological dimension.

We will not yet define either fractal or topological dimensions, but we will note "self similarity" in the following three fractals.

The Cantor Middle-thirds Set. This is perhaps the most simple fractal. We start with the closed interval $[0, 1]$ in the real line and we first take out the middle third $(1/3, 2/3)$, then we are left with the two closed intervals $[0, 1/3]$ and $[2/3, 1]$. Then we repeat the process on each interval, taking out middle thirds and getting more smaller closed intervals. This is how the self similarity arises in the final Cantor Set C.

We can formalize the notion of self-similarity in this case by using *affine transformation*. By applying the function $L(x) = 3x$ to the portion of C in $[0, 1/3]$ we recover C. The same occurs if we apply $R(x) = 3x - 2$ to the portion of C in $[2/3, 1]$ These two functions essentially magnify the two main portions of C. We could further magnify the portions of these portions with other functions, and go on indefinitely. Note magnifications are always powers of $1/3$.

The Sierpinski Triangle. This fractal, which we encountered in the Chaos Game, can be constructed in a manner similar to the Cantor Set C. Start with an equilateral triangle and remove from the middle a triangle with half the big triangle's dimensions rotated 180 degrees. When this is done we are left with three smaller equilateral triangles, and we repeat to each this procedure of removal of a triangle in the middle. The final set, which is an intersection of all the intermediate sets, is non-empty (because at each step the resulting set was a closed subset of the cartesian plane, and the sequence of intermediate sets is nested). This is the Sierpinski triangle.

This set, like the Cantor Set, also has the self-similarity property, but in this case the magnifications are powers of $1/2$. Fractals similar to the Sierpinski triangle can be generated starting with an arbitrary triangle and sequentially removing from the center of each of the triangles a similar triangle with half the dimensions of the one being considered.

The Box Fractal. This is the last fractal being introduced today. To construct it, you start with a square and remove four squares each with a third of the dimensions of the original one centered against each edge of the original square. You are left with five squares you can repeat the procedure to, so you continue indefinitely, and again, the intersection of these sets is the box fractal.