Lecture 3: Bifurcation and the Quadratic Map

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Note that all functions will be from $\mathbb{R}$ to $\mathbb{R}$ unless otherwise noted.

1 Types of cycles and their meaning

Before we proceed into discussing different types of bifurcations, we shall take the time to discuss the concepts of attraction and repulsion, except this time, we will discuss how they relate to cycles instead of points. These concepts will be important in the discussion of bifurcation, because certain types of bifurcation involve the generation of cycles that originate from a fixed point (though this will be defined in more precise detail later).

Definition   An *attracting n-cycle* for a function $F(x)$ is an n-cycle for which $|(F^n)'| < 1$ on the n-cycle, and a *repelling n-cycle* is one for which $|(F^n)'| > 1$ on the n-cycle.

The reader will surely have noticed that the point at which the derivative is to be taken is not specified, however, it is easy to see that $(F^n)' = F' \circ F^{n-1} \cdot F' \circ F^{n-2} \cdot \ldots \cdot F'$, which means that $(F^n)'$ has the same value for any x that is part of the n-cycle since F^a will range over the n-cycle for $0 \le a \le n-1$.

There are two ways of interpreting the meaning of an attracting or repelling n-cycle. One of these ways is as an attracting or repelling point for the function F^n, but another, perhaps more graphically intuitive way of thinking about these cycles is as cycles in which every nth point in the orbit is considered as a sequence that is attracted or repelled from a specific point of the n-cycle. Admittedly, this is not a rigorous mathematical formulation, but should help to provide intuition for the concept.

2 Bifurcations

The notion of a bifurcation is that of a splitting, or division, of significant points to a function by the modification of a parameter that controls the function. More specifically, we will be considering a family of functions $F_\lambda(x)$ parameterized by λ and the behavior of their fixed points and cycles as we vary this parameter. We will find that there are certain critical values of λ for which the structure of the fixed points and cycles of F_λ changes; for critical values with specific properties we will declare that a bifurcation has occurred.

There are two main types of bifurcations we will be discussing:

- Saddle-node bifurcations

- Period-doubling bifurcations

In each of these types of bifurcations, a fixed point "splits" into a fixed point and some other special point or points. The meaning of this will be cleared up momentarily.

2.1 The Saddle-Node bifurcation

Definition A family of functions F_λ is said to undergo a *saddle-node bifurcation* if there exists some value λ_0 and some open interval I that satisfies the following conditions:

1. $\exists \epsilon$ such that $\lambda_0 - \epsilon < \lambda < \lambda_0 \to F_\lambda$ has no fixed points on I.

2. F_{λ_0} has exactly one fixed point on I, and this fixed point is neutral.

3. $\exists \epsilon$ such that $\lambda_0 < \lambda < \lambda_0 + \epsilon \to F_\lambda$ has exactly two fixed points on I, one of which is repelling and one of which is attracting.

Note that the way the definition is formulated it does not matter whether the point "splits" for increasing or decreasing λ since it is always possible to formulate another family of functions $F_{-\lambda}$ which would have the appropriate behavior as $\lambda \to \lambda_0$ if the original family did not.

The definition is fairly clear in giving the intuition for the situation: a fixed point essentially appears in an interval at a specific value of λ and then for $\lambda > \lambda_0$ this fixed point splits into two, one of which is attracting and one of which is repelling.

The reason for the name "Saddle-Node bifurcation" is that one typical situation in which this type of bifurcation arises is in a function with a "saddle", that is, a part in the function that has a slope of exactly 1 at some point and a slope that continuously varies from something less than 1 to something greater than 1, call this function f. If we let $F_\lambda(x) = f(x) + \lambda$, then as $\lambda \to -\infty$, we can have a situation in which the graph of the saddle $y = F_\lambda(x)$ becomes tangent to the line $y = x$ at the point wherein the slope varies through 1. This effect can be seen in Figure 1.

Example: Take $F_c(x) = \frac{1}{1-x} + c$. Solving $F_\lambda(x) = x$ we get:

$$p_\pm = \frac{1}{2}(1 + c \pm \sqrt{c^2 - 2c - 3})$$

At $c = -1$ these two are equal to 0, and then as c drops below -1 we get two distinct fixed points. Furthermore, we have $F_c'(x) = \frac{1}{(1-x)^2}$, which is equal to 1 at $x = 0$, and thus the fixed point for $c = -1$ is neutral. Then when c drops below -1, we can observe that one of the p's becomes slightly less and one becomes slightly more, which will cause F_c' to be less than 1 at one and greater than 1 at the other, that is, we will have an attracting and a repelling fixed point. Thus this is an example of a Saddle-Node bifurcation.

2.2 The Period-Doubling bifurcation

Definition A family of functions F_λ is said to undergo a *period-doubling bifurcation* if there exists some value λ_0 and some union of open intervals I that satisfies the following conditions:
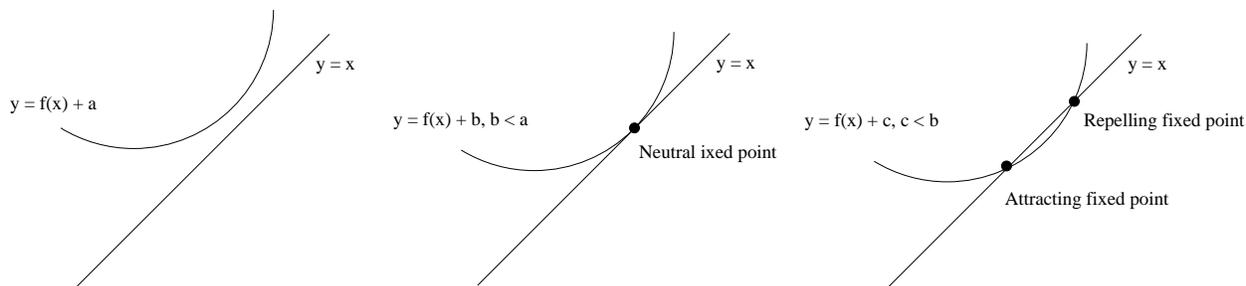
Figure 1: A typical Saddle-Node bifurcation.

1. $\exists \epsilon$ such that $\lambda_0 - \epsilon < \lambda < \lambda_0 \to F_\lambda$ has exactly one attracting (respectively, repelling) n-cycle on I.

2. F_{λ_0} has exactly one n-cycle on I, and this n-cycle is neutral.

3. $\exists \epsilon$ such that $\lambda_0 < \lambda < \lambda_0 + \epsilon \to F_\lambda$ has exactly one 2n-cycle, which is attracting (respectively repelling), and one n-cycle, which is repelling (respectively attracting). Let the 2n-cycle be of the points p_a for $a = 1, 2, \ldots, 2n$.

4. $\lim_{\lambda \to \lambda_0^+} p_a = q(a)$ where each $q(a)$ is a point in the original n-cycle.

The intuition for this type of bifurcation is that we start with some sort of n-cycle, and from this n-cycle we spawn a 2n-cycle, that is, one whose period is doubled, and change the type of n-cycle that we have from attracting to repelling or repelling to attracting.

Example: Take $F_c(x) = x + x^2 + c$. Then we have:

$$
\begin{aligned}
F_c(x) = x + x^2 + c &= x \iff \\
x &= \pm\sqrt{-c}
\end{aligned}
$$

$$
\begin{aligned}
F_c^2(x) &= x^4 + 2x^3 + (2 + 2c)x^2 + (1 + 2c)x + c(2 + c) = x \iff \\
x &= \pm\sqrt{-c}, -1 \pm \sqrt{-1 - c}
\end{aligned}
$$

So for c infinitesimally greater than -1 we have a fixed point for F_c very close to -1, $F_c'(x) = 2x + 1$, and thus the fixed point is attracting. As c gets to -1, this fixed point becomes neutral, since the fixed point moves to $x = -1$, which has $F_{-1}'(-1) = 2(-1) + 1 = -1$. When c becomes infinitesimally less than -1, $\sqrt{-1 - c}$ becomes real, and then we have a 2-cycle $-1 \pm \sqrt{-1 - c}$, since these are each roots of $F_c^2(x) = x$ and not roots of $F_c(x) = x$. $(F_c^2)'(-1 + \sqrt{-1 - c}) = 5 + 4c$, and thus as c becomes infinitesimally less than -1, the two cycle becomes attracting. Furthermore, it is evident that as $c \to -1$, these two fixed points head toward $x = -1$. Thus, since we have a critical value for which an attracting fixed point "splits" into a repelling fixed point and an attracting 2-cycle, we have an example of a Period-Doubling bifurcation.
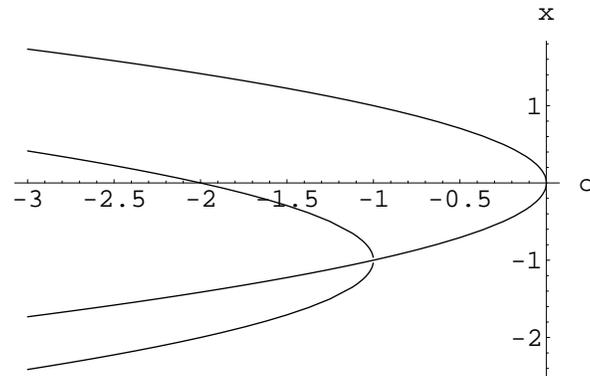
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Figure 2: A bifurcation diagram for the previously considered example $F_c(x) = x + x^2 + c$. In this diagram we can see both the two fixed points for F_c as well as the bifurcation that occurs at $x = -1, c = -1$

2.3 Bifurcation Diagrams

An informative way of presenting some of the bifurcation data we've seen so far is through something known as a "bifurcation diagram". In such a diagram, we plot the paths of fixed points and cycles as we vary the parameter that controls the family of functions. An example is presented in Figure 2.