18.085 Computational Science and Engineering I
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18.085 - Mathematical Methods for Engineers I	Prof. Gilbert Strang

Solutions - Problem Set 2

Section 1.3

7) Suppose A is rectangular (m by n) and
C is symmetric (m by m) matrix

i) $(A^{\mathrm{T}}CA)^{\mathrm{T}}$

$= A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}}$ since $C = C^{\mathrm{T}}$ (symmetric)

$= A^{\mathrm{T}}CA$

$\therefore A^{\mathrm{T}}CA$ is symmetry $_{\#}$

$(A^{\mathrm{T}})_{n\times m}\ (C)_{m\times m}\ (A)_{m\times n}$

$\therefore A^{\mathrm{T}}CA$ is $(n \times n)$ $_{\#}$

ii) Let $A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & a_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

$$A^{\mathrm{T}}A = \begin{bmatrix} \cdots\ a_1^{\mathrm{T}}\ \cdots \\ \cdots\ a_2^{\mathrm{T}}\ \cdots \\ \cdots\ a_n^{\mathrm{T}}\ \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ a_1 & a_2 & a_n \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_1^{\mathrm{T}}a_1 & \cdots & \cdots & \cdots \\ \vdots & a_2^{\mathrm{T}}a_2 & & \\ \vdots & & \ddots & \\ \cdots & & & a_n^{\mathrm{T}}a_n \end{bmatrix}$$

Since $a_i^{\mathrm{T}}a_i = a_i^2 \geq 0$, we conclude that

$A^{\mathrm{T}}A$ has no negative numbers on its diagonal $_{\#}$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_{L^{\mathrm{T}}} \quad \#$$

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}}_{L^{\mathrm{T}}} \quad \#
\end{aligned}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{L^{\mathrm{T}}}
\end{aligned}
$$

Section 1.4

3) $-u'' = \delta(x - 1/3) + \delta(x - 2/3)$

General Fixed-Fixed Solution

$$
u = \begin{cases} (1-a)x & , \; x \le a \\ (1-x)a & , \; x \ge a \end{cases}
$$

$$
-u'' = \delta(x - 1/3)
$$

$$
u = \begin{cases} (1-1/3)x & , \; x \le 1/3 \\ (1-x)1/3 & , \; x \ge 1/3 \end{cases}
$$

$$
-u'' = \delta(x - 2/3)
$$

$$
u = \begin{cases} (1-2/3)x & , \; x \le 2/3 \\ (1-x)2/3 & , \; x \ge 2/3 \end{cases}
$$

Combining two single-load solutions:

$$
u = \begin{cases} (1-1/3)x + (1-2/3)x & , \; x \le 1/3 \\ (1-x)1/3 + (1-2/3)x & , \; 1/3 \le x \le 2/3 \\ (1-x)1/3 + (1-x)2/3 & , \; x \ge 2/3 \end{cases}
$$

$$
= \begin{cases} x & \text{for } \; x \le 1/3 \\ 1/3 & \text{for } \; 1/3 \le x \le 2/3 \\ 1-x & \text{for } \; x \ge 2/3 \end{cases} \quad \#
$$

Second Method

$$u(x) = \begin{cases} Ax+B & , \ x \le 1/3 \\ Cx+D & , \ 1/3 \le x \le 2/3 \\ Ex+F & , \ x \ge 2/3 \end{cases}$$

$$u(0) = 0$$

$$A \cdot 0 + B = 0$$
$$\therefore B = 0_{\#} \quad \text{———} \ 1$$

$$A(1/3) + \cancel{B}^{\,0} = C(1/3) + D$$
$$A = C + 3D \quad \text{———} \ 2$$
$$A - 1 = C \quad \text{———} \ 3$$

$$C(2/3) + D = E(2/3) + F$$
$$2C + 3D = 2E + 3F \quad \text{———} \ 4$$
$$C - 1 = E \quad \text{———} \ 5$$

$$E(1) + F = 0$$
$$E = -F \quad \text{———} \ 6$$

$2 \rightarrow 3$

$$\cancel{C} + 3D - 1 = \cancel{C}$$
$$D = \frac{1}{3}_{\#}$$

$6 \rightarrow 4$

$$2C + 3(1/3) = 2E + 3(-E)$$
$$2C + 1 = -E \quad \text{———} \ 7$$

$7 - 2 \times 5$

$$1 + 3 = -3E$$
$$E = -1_{\#}$$

$$C = 1 + E$$
$$= 1 + (-1)$$
$$\therefore C = 0_{\#}$$

$$E = -F$$
$$\therefore F = 1_{\#}$$

$$A - 1 = C$$
$$A = 1 + 0$$
$$\therefore A = 1_{\#}$$

$$\therefore u(x) = \begin{cases} x & , \ x \le 1/3 \\ 1/3 & , \ 1/3 \le x \le 2/3 \\ 1-x & , \ x \ge 2/3 \end{cases}$$

5) Free-Free condition

$$u(x) = -R(x-a) + Cx + D$$
$$u'(0) = 0 \qquad\qquad u'(1) = 0$$
$$u'(0) = 0 + C = 0$$
$$\therefore C = 0$$
$$u'(1) = -1 + C = 0$$
$$\therefore C = 1$$

$\therefore$ There are no solutions for C and D

C cannot be 0 and 1 at the same time #

7) $f(x) = \delta(x-1/3) - \delta(x-2/3)$

$u'(0) = 0, \qquad u'(1) = 0$

$$u(x) = \begin{cases} Ax + B & , \ x \leq 1/3 \\ Cx + D & , \ 1/3 \leq x \leq 2/3 \\ Ex + F & , \ x \geq 2/3 \end{cases}$$

$$u'(0) = A = 0 \ \#$$
$$\cancel{A(1/3)}^{0} + B = C(1/3) + D$$
$$3B = C + 3D \quad \text{——} \ 2$$
$$A - 1 = C$$
$$\therefore C = -1 \ \#$$
$$C(2/3) + D = E(2/3) + F$$
$$2C + 3D = 2E + 3F \quad \text{——} \ 3$$

$$\left.\begin{aligned} C + 1 &= E \\ E &= 0 \ \# \\ \frac{d}{dx}(Ex+F) &= 0 \\ E &= 0 \end{aligned}\right\} \text{redundant, less one equation}$$

From 3

$$2C + 3D = 3F$$
$$3D = 3F - 2C$$
$$= 3F + 2$$
$$D = F + 2/3$$

From 2

$$3B = -1 + 3D$$
$$B = \frac{-1 + 3F + 2}{3}$$
$$= F + 1/3$$

$$u(x) = \begin{cases} F + 1/3 & , \ x \leq 1/3 \\ -x + F + 2/3 & , \ 1/3 \leq x \leq 2/3 \\ F & , \ x \geq 2/3 \end{cases}$$

F can take any value $F \in \mathbf{R}$

$\therefore$ infinitely many solutions for $u(x)_{\#}$

12) $u'''' = \delta(x)$, $C(x) = \begin{cases} 0 & , \ x \leq 0 \\ \dfrac{x^3}{6} & , \ x \geq 0 \end{cases}$

Cubic spline $C(x)$ is a particular solution for u

$u(x) = C(x) + Ax^3 + Bx^2 + Gx + D$

Given that $u(1) = 0 \qquad u''(1) = 0$
$u(-1) = 0 \qquad u''(-1) = 0$

$$u(1) = \frac{1}{6} + A + B + G + D = 0$$

$$A + B + G + D = -\frac{1}{6} \quad \text{——} 1$$

$$u(-1) = 0 + (-A) + B - G + D = 0$$

$$A - B + G - D = 0 \quad \text{——} 2$$

$$u''(1) = 1 + 6A(1) + 2B = 0$$

$$6A + 2B = -1 \quad \text{——} 3$$

$$u''(-1) = 0 - 6A + 2B = 0$$

$$\therefore B = 3A$$

$$A = -\frac{1}{12}_{\#} \qquad B = -\frac{1}{4}_{\#}$$

$$-\frac{1}{12} + \left(-\frac{1}{4}\right) + G + D = -\frac{1}{6}$$
$$-\frac{1}{12} - \left(-\frac{1}{4}\right) + G - D \quad = 0$$

$$G + D = \frac{1}{6}$$

$$G - D = -\frac{1}{6}$$

$$\therefore G = 0 \quad , \quad D = \frac{1}{6}_{\#}$$

$$\therefore u(x) = C(x) - \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6}_{\#}$$

Section 1.5

1) $K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{aligned} Q\Lambda Q^{\mathrm{T}} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\ &= K \ \# \end{aligned}$$

4) $>> K = \mathsf{toeplitz}([2 \ \ -1 \ \ \mathsf{zeros}(1,3)]);$

$>> [Q, E] = \mathsf{eig}(K);$

$>> [DST = Q * \mathsf{diag}([-1 \ \ -1 \ \ \ 1 \ \ -1 \ \ \ 1])$

$DST =$

$$\begin{array}{rrrrr} 0.2887 & 0.5000 & 0.5774 & 0.5000 & 0.2887 \\ 0.5000 & 0.5000 & -0.0000 & -0.5000 & -0.5000 \\ 0.5774 & -0.0000 & -0.5774 & 0.0000 & 0.5774 \\ 0.5000 & -0.5000 & -0.0000 & 0.5000 & -0.5000 \\ 0.2887 & -0.5000 & 0.5774 & -0.5000 & 0.2887 \end{array}$$

$>> JK = [1{:}5]' * [1{:}5];$

$>> \sin(JK * \mathsf{pi}/6)/\mathsf{sqrt}(3)$

ans =

$$\begin{array}{rrrrr} 0.2887 & 0.5000 & 0.5774 & 0.5000 & 0.2887 \\ 0.5000 & 0.5000 & 0.0000 & -0.5000 & -0.5000 \\ 0.5774 & 0.0000 & -0.5774 & -0.0000 & 0.5774 \\ 0.5000 & -0.5000 & -0.0000 & 0.5000 & -0.5000 \\ 0.2887 & -0.5000 & 0.5774 & -0.5000 & 0.2887 \end{array}$$

$DST = \sin(JK * \mathsf{pi}/6)/\mathsf{sqrt}(3)$

$>> DST'$

ans =

$$\begin{matrix} 0.2887 & 0.5000 & 0.5774 & 0.5000 & 0.2887 \\ 0.5000 & 0.5000 & -0.0000 & -0.5000 & -0.5000 \\ 0.5774 & -0.0000 & -0.5774 & -0.0000 & 0.5774 \\ 0.5000 & -0.5000 & 0.0000 & 0.5000 & -0.5000 \\ 0.2887 & -0.5000 & 0.5774 & -0.5000 & 0.2887 \end{matrix}$$

$>> \mathsf{inv}(DST)$

ans =

$$\begin{matrix} 0.2887 & 0.5000 & 0.5774 & 0.5000 & 0.2887 \\ 0.5000 & 0.5000 & -0.0000 & -0.5000 & -0.5000 \\ 0.5774 & -0.0000 & -0.5774 & -0.0000 & 0.5774 \\ 0.5000 & -0.5000 & -0.0000 & 0.5000 & -0.5000 \\ 0.2887 & -0.5000 & 0.5774 & -0.5000 & 0.2887 \end{matrix}$$

$DST' = \mathsf{inv}(DST)$

7) $C_4 = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$

$[Q, E] = \mathsf{eig}(C_4)$

$$Q = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 0 & -1/2 \\ 1/2 & 0 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 0 & -1/2 \\ 1/2 & 0 & -1/\sqrt{2} & 1/2 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & & 0 & \\ & 2 & & \\ & & 2 & \\ & 0 & & 4 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$f_1 = a_1\, q_1 + a_2\, q_2 + a_3\, q_3 + a_4\, q_4$

$f_2 = b_1\, q_1 + b_2\, q_2 + b_3\, q_3 + b_4\, q_4$

$f_3 = c_1\, q_1 + c_2\, q_2 + c_3\, q_3 + c_4\, q_4$

$f_4 = d_1\, q_1 + d_2\, q_2 + d_3\, q_3 + d_4\, q_4$

$$[f_1 \quad f_2 \quad f_3 \quad f_4] = [q_1 \quad q_2 \quad q_3 \quad q_4] \underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}}_{\text{denote by } A}$$

f_i and q_i are the column i of their respective matrix

$F = QA$

$Q^{-1}QA = Q^{-1}F$

$A = Q^{-1}F$

From MATLAB

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & 0 & -2 & 0 \end{bmatrix} \#$$

9) $$\Delta_- = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}_{4\times 3}$$

${\Delta_-}^{\mathrm{T}}\Delta_-$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = K_3$$

#

$\Delta_-{\Delta_-}^{\mathrm{T}}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = B_4 \quad \#$$

$$\mathsf{eig}(K_3) = \begin{bmatrix} 0.5858 \\ 2.0000 \\ 3.4142 \end{bmatrix}$$

$$[\,u \quad s \quad v\,] = \mathsf{svd}(\Delta_-)$$

$$s = \begin{bmatrix} 1.8478 & & \\ & 1.4142 & \\ & & 0.7654 \end{bmatrix}$$

$$\sigma_1^2 = 1.8478^2 = 3.4142 = \lambda_3$$
$$\sigma_2^2 = 1.4142^2 = 2 = \lambda_2$$
$$\sigma_3^2 = 0.7654^2 = 0.5858 = \lambda_1$$

$\therefore$ The eigenvalues of K_3 are the squared singular values σ^2 of Δ_- #