PROFESSOR: Hi, welcome back to recitation. In the lecture, we've learned very important concepts-- linear space and linear subspace. Well, as you can imagine, if we call something a space, we're putting a lot of things, a lot of objects into one set. But for linear space, we want to put them in according to a particular manner. So can you recall what are the conditions for a set to be a linear space?

You take any two elements from that set, take the sum of them. You want the sum to be still in the same set. That's the first condition.

Second, you take any multiple of any element from that set, the result will still be in that set. That's the second condition. And if, within a linear space, you can find a subset which also satisfies the two conditions, that will give you a subspace. Today, we're going to look at this example to review these two important properties of linear space and subspace.

I have two vectors, x_1 and x_2. Both of them are vectors in R^3. So, as you can see, I've drawn them here. This is x_1, and this is x_2. So first, we want to find the subspace generated by x_1. I call it V_1. Let me say a word about this "generated by."

So what do I mean by a subspace generated by x_1? I'm looking for the smallest subspace that contains x_1, as small as you can get. Similarly, I want to find out the subspace generated by x_2, call it V_2. Then we want to say something about the intersection of V_1 and V_2. That's the first question.

And the second question, we want to put vector x_1 and x_2 together and look at the subspace generated by x_1 and x_2 at the same time. So I call it V_3. And a good question to be asked here is: what is the relation of v3 to V_1 union V_2? Do you think they're equal?

Within the second question, I would also like you to find a subspace, call it S, of V_3 such that neither x_1 nor x_2 is in s. And the last question is I'd like you to say something about the intersection of V_3 with the xy-plane. So notice that, of course, xy-plane is also a subspace of R^3. So again, I'm looking at the intersection of two subspaces.

All right, why don't you hit the pause now, and try to solve these three problems on your own. And I'd like you to identify your answers in this picture whenever you can. I'll come back later and continue working with you.

OK, how did your drawing go? Let's look at this together. First, we want to find subspace generated by x_1. So here is x_1. Let's keep in mind the two conditions that a subspace has to satisfy. Well, if I want to obtain the subspace, at least I have to be able to take any multiple of x_1, right? So that means at least I have to include the straight line that contains x_1.

So I'm going to try to draw the straight line here. So I'm just simply going to extend this x1 along to both directions. So x_1, I will try very hard to make it straight. But to be honest, it's really hard for me to draw straight lines on the board. Is that straight? Seems fine.

All right, so this entire line contains x_1. So at least this line has to be in V_1. Is there anything else beyond this line?

Now let's turn to the second condition. The second condition says that we have to be able to take any two elements, take the sum, and the sum will remain in that set. Does it work for this line? You take any two vectors on this line, or you can say any two points on this line, and then you take to sum of them.

Of course, you still get something on this line. You won't be able to escape from it. Which means this line is a perfect set that satisfies the two conditions. So this line simply gives me V_1. That is the smallest subspace that contains x_1. So in other words, the subspace generated by x_1 is V_1.

So similarly, let's look at x_2. What is the subspace generated by x_2? Again, you get the entire straight line that contains x_2. So I'm going to extend x_2 in both directions. I hope it's straight. Not too bad. That will give me V_2. All right, V_1, V_2. Both of them are subspaces of R^3.

Now let's look at the intersection of V_1 and V_2. So we know that both of them are straight lines. And clearly they're not parallel, because x_1 and x_2 are not parallel. So what is the intersection of V_1 and V_2? The intersection of V_1 and V_2 is the only point at which they cross. And where's that point? It's here, right at the origin. Because both of them pass the origin. I'm going to use O to denote. That's the intersection of V_1 and V_2.

This is a set with only one element. What can you say about this set? I claim this is also a subspace of R^3. By saying space, we usually mean a lot of objects together. But look at this set. This set fits perfectly into the conditions of being a linear space. You take any multiple of 0, again you get 0-- 0 plus 0, you get 0. So that's a perfectly fine subspace of R^3.

All right, so what we have got here is: I take the intersection of V_1 and V_2, and the result, again, becomes a subspace, which only contains the origin. That completes the first question.

Let's look at the second one. In the second question, I want to put x_1 and x_2 together, and look at the subspace generated by x_1 and x_2. And I would also like you to say something about the relation between V_1 union V_2 to the subspace generated by x_1, x_2. Let's try to answer the second question first. Is there a chance that V_1 union V_2 equal to v3? OK, so what is V_1 union V_2? That's clearly just two lines, right? This line union this line.

Is there a chance that this union will be a subspace? Let's check the two conditions. First, you take any multiple of the elements in this union. It's either on this line or on this line. Seems that the multiple is still going to stay inside the union. So the first condition is actually satisfied.

What about the second one? The second one says that I have to be able to take any sum, the sum of any two elements, from these two lines. Let's just try a simple sum, x_1 plus x_2. So what is x_1 plus x_2? You just sum up each coordinate. That will give you [2, 5, 3]. In this picture-- can I draw it in this picture? It's going to be somewhere here. That's x_1 plus x_3.

Did you notice something? You clearly have got out of this union. So this sum is not inside V_1 union V_2. Which means V_1 union V-2 is not a subspace. Then it's impossible that this union will equal to V_3. So the answer to the second question is no, V_3 is not equal to V_1 union V_2.

Now let's identify V_3. Well, as you can see, since we have seen from this argument that we have to be able to include this diagonal vector here. But in fact, as you can see since we can take any elements from these two lines, we're actually including every vector on the plane spanned by V_1 and V_2. So in other words, I'm actually looking at this huge plane that is spanned by V_1 and V_2. That will give me V_3. So that's reasonable. I'm looking at the subspace generated by two lines in R^3. And that two line will be able to span a plane in R^3.

Now for the last part of question two, I want to find a subspace S of V_3-- so I want to find a subspace of this plane-- such that x_1 is not S, x_2 is not in S either. So I want to stay away from this line and this line. Can you find such a subspace of V_3? It's right here. Because if you look at this vector, x_1 plus x_2-- sorry, that should be x_2. If you look at this vector, and if you look at the subspace generated by this vector, again you know it's going to be a line. And this line is right here.

This line forms a perfect subspace of V_3. But neither x_1 nor x_2 is inside this subspace. So let's just make it S. Of course, the choice is not unique. You can take twice x_1 plus x_2 or x_1 plus twice x_2. OK, we have completed the second problem.

The last problems ask us to find the intersection of V_3 with xy-plane. Well, just think about that. We've identified V_3 as this plane spanned by line V_1 and V_2. And xy is also a plane. So we're talking about the intersection of two planes in R^3. What would that be? So you have two planes intersect. The intersection will be a straight line again, right? So let's locate that straight line.

We want to find something that is inside V_3 and xy-plane at the same time. What can you say about the points in xy-plane? The z-coordinate has to be 0. And at the same time, we know that at least x_1 and x_2 are in V_3. So did you notice that? x_2 is a vector that lies in V_3. But at the same time, the z-coordinate of x_2 is 0. That is what we're looking for.

So the intersection of V_3 with xy-plane will simply be the line that contains x_2. And here we've identified that as V_2. That's it. This is a subspace of R^3. And this is the subspace of R^3. The intersection is again a subspace of R^3.

I hope you've learned a way to somehow visualize this linear space and subspace through this exercise. Thank you for watching, and I'll see you next time.