18.099 Final Project # 5
The Cartan matrix of a Root System

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July 30, 2004

A root system in a Euclidean space V with a symmetric positive definite inner product $\langle , \rangle$ is a finite set of elements Δ of V such that

1. Δ spans V;
2. for every root $\alpha \in \Delta$ and every $\beta \in \Delta$, $\beta - \frac{2\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$ is a root. Moreover, every root has such an expansion;
3. the number $\frac{2\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}$ is an integer for all $\alpha, \beta \in \Delta$.

If $2\alpha \notin \Delta$ for all $\alpha \in \Delta$, the root system Δ is reduced. For every root system in V, there exists a simple root system $\Pi \subset V$, such that

1. the elements of Π form a basis for V;
2. every root $\beta \in \Delta$ is a linear combination $\sum_{\pi_i \in \Pi} c_i \pi_i$ with every c_i being of the same sign.

If the coefficients c_i for a root β are all nonnegative, the root is positive. Otherwise, it is negative. The set of positive roots in Δ is denoted as Δ^+.

Definition 1 *Let $\Pi \in \Delta$ be a simple root system in V, and let the elements of Π be enumerated as $\{\alpha_i\}_{i=1}^n$ where n is the dimension of V. The Cartan matrix A is the square matrix given by*

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}.$$

It is obvious that this matrix is dependent on the enumeration of the elements of Π. However, the Cartan matrices of a root system with different enumerations of the simple roots are related by permutation matrices. To prove this, an important fact about permutation matrices is necessary

Theorem 2 *Let P^{ij} be the identity matrix with rows i and j switched. For any square matrix B, the matrix $B' = P^{ij}B(P^{ij})^{-1}$ is B with rows and columns i and j switched.*

Proof: Because P^{ij} is the identity matrix with rows i and j switched, the left-hand product $P^{ij}B$ is the matrix B^r, which is B with rows i and j switched. The transpose $(B^r)^\top = B^\top(P^{ij})^\top$ is the transpose of B with columns i and j switched. Since P^{ij} is a simple permutation matrix (i.e., it is only one row exchange away from the identity matrix), $(P^{ij})^{-1} = (P^{ij})^\top$. Therefore, $(B^r)^\top = B^\top(P^{ij})^{-1}$. Multiplying $(B^r)^\top$ on the left by the same P^{ij} will again switch rows i and j. This results in the matrix $B'' = P^{ij}(B^r)^\top = P^{ij}B^\top(P^{ij})^{-1}$, which is the transpose of B with the i-th and j-th rows and columns switched. Its transpose, $B' = (B'')^\top = P^{ij}B(P^{ij})^{-1}$, is therefore B with the i-th and j-th rows and columns switched. □

With this theorem, it is now possible to relate the Cartan matrices given by different enumerations of the same reduced root system.

Corollary 3 *Let A be the Cartan matrix of a reduced root system (Π, Δ) with a fixed enumeration of the simple roots $\{\alpha_i\}_{i=1}^n$ and let A' be the Cartan matrix of the same root system with the same enumeration of the simple roots, except that roots α_k and α_l are reversed. Then $A' = P^{kl}A(P^{kl})^{-1}$.*

Proof: The Cartan matrix for the first enumeration is given by

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

For all entries (i, j), $i, j \neq k$ and $i, j \neq l$, the Cartan matrix $A'_{ij} = A_{ij}$. All entries A_{kj} are given by $\frac{2\langle \alpha_l, \alpha_j \rangle}{\langle \alpha_l, \alpha_l \rangle}$, and the entries A_{lj}, A_{ik}, A_{il} are given in a similar manner. Thus, the entries of A' involving k are switched with those involving l and vice versa. Then A' is A with rows and columns i and j switched. By Theorem 2, $A' = P^{ij}A(P^{ij})^{-1}$. □

Because any enumeration of a set of simple roots is related to any other enumeration by some finite number of permutations, the relationship between any two Cartan matrices for a root system is an isomorphism by the conjugate product of permutation matrices.

Definition 4 *Two Cartan matrices are isomorphic if they are conjugate by a product of permutation matrices.*

Here are some examples of Cartan matrices.

Example 5 *The Cartan matrix of the root system of type A_n as defined in Final Project 3 is the $n \times n$ tridiagonal matrix with 2's on the main diagonal and -1's on the upper and lower diagonals.*

Proof: Let the root system of type A_n be as defined in [3]. The simple roots Π are enumerated as $\{e_i - e_{i+1}\}_{i=1}^{n+1}$ so that there are n distinct simple roots. For any two roots $e_i - e_{i+1}$ and $e_{i+1} - e_{i+2}$, the appropriate entry of A is

$$\frac{2\langle e_i - e_{i+1}, e_{i+1} - e_{i+2}\rangle}{\langle e_i - e_{i+1}, e_i - e_{i+1}\rangle}.$$

The numerator of that fraction is

$$\langle e_i, e_{i+1}\rangle - \langle e_i, e_{i+2}\rangle - \langle e_{i+1}, e_{i+1}\rangle + \langle e_{i+1}, e_{i+2}\rangle$$

and since all of the e's are orthonormal to each other and of length 1, the numerator reduces to -1. The denominator is

$$\langle e_i - e_{i+1}, e_i - e_{i+1}\rangle = \langle e_i, e_i\rangle - \langle e_i, e_{i+1}\rangle - \langle e_{i+1}, e_i\rangle + \langle e_{i+1}, e_{i+1}\rangle$$

which reduces to 2 for the same reasons. Hence, the value of an entry directly above or below the diagonal in this Cartan matrix is always -1. The diagonal entries are always 2, because the inner products in the numerator and denominator are identical. All other entries are 0 because the inner product in the numerator involes no identical terms. Therefore, the Cartan matrix A of the root system of type A_n is given by

$$A_{ij} = \begin{cases} 0 & |i-j| > 1, \\ -1 & |i-j| = 1, \\ 2 & |i-j| = 0. \end{cases}$$

□

An interesting fact about this Cartan matrix is that it is identical to the stiffness matrix of a system with $n-1$ springs with unit spring constants and n unit masses.

Example 6 *Let the root system of the type B_4 be defined as in [1]. The simple roots Π are $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4\}$. The positive roots Δ^+ are $\{e_i \pm e_j\}_{i<j} \cup \{e_i\}_{i=1}^4$. The Cartan matrix is*

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

Proof: These roots form a basis for $\mathbb{R}^4$. For any vector $v \in \mathbb{R}^4 = \sum_{i=1}^4 c_i e_i$, let $c_1 = b_1, c_2 = b_2 - b_1, c_3 = b_3 - b_2$, and $c_4 = b_4 - b_3$. The sum then expands to $b_1(e_1 - e_2) + b_2(e_2 - e_3) + b_3(e_3 - e_4) + b_4 e_4$. It is clear, then, that the simple roots in B_4 are a basis for $\mathbb{R}^4$ and thus they span it. Additionally, every $\delta \in B_4$ has an integral expansion in the simple roots. Recall that $B_4 = \{e_i \pm e_j\}_{i \neq j} \cup \{\pm e_i\}_{i=1}^4$. Let each simple root enumerated above be denoted by π_i. The roots $\pm e_i$ can be expressed as the sum $\pm \sum_{j=1}^4 \pi_j$. Therefore, the roots $\pm e_i \mp e_j$ can be expressed as the sum $\pm \sum_{k=1}^4 \pi_k + \mp \sum_{l=1}^4 \pi_l$. In this way, all roots are combinations of the simple roots with integer coefficients of the same sign. The positive roots are those with all positive coefficients in their expansion across the simple roots. Clearly every e_i is a positive root, and thus every sum $e_i + e_j, j \neq i$ is also a positive root. Because every simple root but the last one is identical to the simple roots of A_4, the Cartan matrix is identical except in the last sub-diagonal entry, where it is -2. This is because the square of the length of e_4 is half of the square of the length of $e_3 - e_4$. $\square$

The properties of all Cartan matrices are summarized here:

Theorem 7 *Any Cartan matrix A of a root system (Π, Δ) has the following properties.*

1. *every entry is an integer;*

2. *all diagonal entries are 2;*

3. *all off diagonal entries are non-positive;*

4. $A_{ij} = 0$ *if and only if* $A_{ji} = 0$*;*

5. there exists a diagonal matrix D *with positive entries such that* DAD^{-1} *is symmetric positive definite.*

Proof: 1. This is true by property 3 of the definition of an abstract root system [1].

2. Every entry $A_{ii} = \frac{2\langle\alpha_i,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} = 2$.

3. Let α_i and α_j be two distinct simple roots. By property 3 of root systems, $\alpha_i - \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j}\alpha_j$ is a root α_k. Since α_i and α_j are simple, and α_k is a linear combination of the two, property 2 of simple roots requires that the number $\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j}$ is negative.

4. Suppose $A_{ij} = 0$. Then $\langle\alpha_i, \alpha_j\rangle = 0$. By the reflexivity of the inner product, $\langle\alpha_i, \alpha_j\rangle = \langle\alpha_j, \alpha_i\rangle$. Therefore, if the numerator of $A_{ij} = 0$, the numerator of A_{ji} must also be 0 and vice versa.

5. Let $D = diag(|\alpha_1|, |\alpha_2|, ..., |\alpha_n|)$ where $|\alpha_i| = \langle\alpha_i, \alpha_i\rangle^{\frac{1}{2}}$. Then $DAD^{-1}_{ij} = A_{ij}\frac{\langle\alpha_i,\alpha_i\rangle^{\frac{1}{2}}}{\langle\alpha_j,\alpha_j\rangle^{\frac{1}{2}}}$. Because $A_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$ this product reduces to $DAD^{-1}_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle^{\frac{1}{2}}\langle\alpha_j,\alpha_j\rangle^{\frac{1}{2}}}$ which, by the reflexivity of the inner product, is clearly symmetric. Denote the j-th entry of the simple root α_i in some basis of V as b_{ij}. The matrix B defined by $B_{ij} = \frac{b_{ij}}{\langle\alpha_i,\alpha_i\rangle^{\frac{1}{2}}}$ gives a Cholesky factorization for A because $B^\top B_{ij}$ is the inner product of α_i with the α_j divided by the lengths of α_i and α_j. This is precisely DAD^{-1}_{ij}. Because the simple roots form a basis for V, B is invertible. Thus A is positive definite.

□

Example 8 *There are* 5 *reduced root systems in* $\mathbb{R}^2$*:* $A_1 \oplus A_1$*,* A_2*,* B_2*,* C_2*, and* G_2*:*

1. The root system $A_1 \oplus A_1$ consists of $\{\pm e_1, \pm e_2\}$ so that the angle between the simple roots $\{e_1, e_2\}$ is $\frac{\pi}{2}$ and each simple root has unit length. The off diagonal entries of the Cartan matrix must be 0 because the simple roots are orthogonal. Therefore, the Cartan matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

and since the Cartan matrix is already symmetric and positive definitie, the diagonalizing matrix is the identity matrix.

2. The root system A_2 consists of 6 vectors arranged in a hexagon-like fashion and the angle between the two simple roots is $\frac{\pi}{3}$, each of unit length. The Cartan matrix of this root system is thus $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and since the Cartan matrix is already symmetric and positive definite, the diagonalizing matrix is the identity matrix.

3. The simple roots of the root system B_2 are $\{e_1 - e_2, e_2\}$, resulting in an angle of $\frac{3\pi}{4}$ and root lengths of 1 and $\sqrt{2}$. Thus the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$. The diagonalizing matrix is therefore $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$. The product $DAD^{-1} = \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$ has determinant 2 so DAD^{-1} is symmetric positive definite.

4. The simple roots of the root system C_2 are $\{e_2 - e_1, 2e_1\}$. Thus, the angle between them is $\frac{3\pi}{4}$ and their lengths are $\sqrt{2}$ and 2. The Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ and the diagonalizing matrix D is $\begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ so that the product $DAD^{-1} = \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$ is symmetric positive definite. Note that the diagonalizing matrix for C_2 is just $\sqrt{2}$ times the diagonalizing matrix for B_2. In that sense, these root systems are equivalent.

5. The simple roots of the root system G_2 have lengths $\sqrt{3}$ and 1 and the angle between them is $\frac{5\pi}{6}$. Therefore, the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, the diagonalizing matrix is $\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}$, and the symmetric positive definite product of the two is $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$.

Theorem 9 *A reduced root system is reducibile if and only if for some choice of a simple root system and some enumeration of the indices of the simple roots, the Cartan matrix is block diagonal with more than one block.*

Proof: Suppose that a root system is reducible as $\Delta = \Delta' \cup \Delta''$ so that any $\delta' \in \Delta'$ is orthogonal to any $\delta'' \in \Delta''$. Let the simple roots of Δ be $\{\alpha_i\}_{i=1}^n$ so that $\{\alpha_i\}_{i=1}^r \in \Delta'$ and $\{\alpha_i\}_{i=r+1}^n \in \Delta''$. Then for any $i \in \{1, 2, .., r\}$ and $j \in \{r+1, r+2, ..., n\}$ (i.e., i and j represent roots from different parts of Δ's decomposition), $A_{ij} = 0$. Then clearly A is block diagonal with 2 blocks. If a root system is reducible into n orthogonal components, by the same argument, it will have n blocks.

Now, assume that the Cartan matrix of a root system Δ is block diagonal with 2 blocks. Let the roots corresponding to the first block be $\{\alpha_i\}_{i=1}^r$ and those in the second block be $\{\alpha_i\}_{i=r+1}^n$. Assume α is a positive root in Δ. By theorem 7 of [2], $\alpha = a_1 + a_2 + ... + a_k$, where each a_i is a simple root, and repetitions are allowed. Furthermore, each partial sum $a_1 + a_2 + ... + a_c$ is a root, for all $c \leq k$. Let B be the c-th partial sum for α and let A be the next simple root in the sum for α. Then B and $B + A$ are both roots. Theorem 10 of [1] requires that for all n such that $B + nA$ is a root or 0, there exists positive p and q such that $-p \leq n \leq q$ and $p - q = \frac{2\langle B,A\rangle}{\langle B,B\rangle}$.

Suppose the roots that sum to B and A are in different blocks, meaning that $\langle B, A\rangle = 0$ and $p = q$. Then if $B + A$ is a root ($n = 1$ in the theorem from [1]), $B - A$ must also be a root. However, this results in a root which is a sum of simple roots with mixed sign coefficients. This contradicts the definition of simple roots. Therefore, the only such p that does not result in this contradiction is $p = 0$. This means that in any expansion of a positive root in a root system whose Cartan matrix has 2 blocks, all of the coefficients for the set of simple roots in one orthogonal component must be 0.

□

If there is only one block, the root system is said to be irreducible.

Theorem 10 *The matrix D which diagonalizes the Cartan matrix of a root system is determined uniquely up to a scalar multiple on each block of A.*

Proof: For every pair α_i and α_j that are in different blocks, the entry A_{ij} of the Cartan matrix is 0. Therefore, any diagonal matrix D symmetrizes such entries in the Cartan matrix A. For any matrix $D' = diag(a_1, a_2, ..., a_n)$ that symmetrizes A, and any two simple roots α_i and α_j which are in the same block, $D'A(D')^{-1}_{ij} = A_{ij}\frac{a_i}{a_j}$. Because D' symmetrizes A, $A_{ij}\frac{a_i}{a_j} = A_{ji}\frac{a_j}{a_i}$. This reduces to $\frac{a_i}{|\alpha_i|^2 a_j} = \frac{a_j}{|\alpha_j|^2 a_i}$ and finally to $a_i^2 = \frac{|\alpha_i|^2}{|\alpha_j|^2}a_j^2$. This implies that a_i and a_j are determined uniquely up to a scalar multiple by the relative lengths of α_i and α_j. □

With this theorem, it is clear that for any irreducible root system, the matrix D gives the relative lengths of the roots. By definition, the entries of the Cartan matrix A give the angles between any two simple roots. Then, A and D completely characterize the simple roots.

Corollary 11 *Any Cartan matrix determines a set of simple roots uniquely up to a scalar multiple of an orthogonal transformation on each irreducible component*

Proof: From Theorem 10 it is clear that given a Cartan matrix A, there is a diagonalizing matrix D which is unique up to a scalar multiple in each block of A. Thus, A itself determines the relative lengths of some set of simple roots (up to a scalar multiple on each block). Additionally, every entry (i, j) of A gives the angle between α_i and α_j. So for each irreducible component of A, there is a unique root system. $\square$

Example 12 *The root system $A_1 \oplus B_2$ has $\{e_1, e_2 - e_3, e_3\}$ as simple roots. It's Cartan matrix is*

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

which is block diagonal with the Cartan matrix for A_1 in the first block, and the Cartan matrix for B_2 in the second.

References

[1] Dr. Anna Lachowska, *Final Project 2: Abstract Root Systems*, preprint, MIT, 2004.

[2] Dr. Anna Lachowska, *Final Project 4: Properties of Simple Roots*, preprint, MIT, 2004.

[3] Dr. Anna Lachowska, *Final Project 3: Properties of Simple Roots*, preprint, MIT, 2004.