Joint Distributions, Independence
Covariance and Correlation
18.05 Spring 2014

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Joint Distributions

X and Y are jointly distributed random variables.

Discrete: Probability mass function (pmf):

$$p(x_i, y_j)$$

Continuous: probability density function (pdf):

$$f(x, y)$$

Both: cumulative distribution function (cdf):

$$F(x, y) = P(X \leq x, Y \leq y)$$

Discrete joint pmf: example 1

Roll two dice: $X = \#$ on first die, $Y = \#$ on second die

X takes values in 1, 2, ..., 6, Y takes values in 1, 2, ..., 6

Joint probability table:

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

pmf: $p(i, j) = 1/36$ for any i and j between 1 and 6.

Discrete joint pmf: example 2

Roll two dice: $X = \#$ on first die, $T =$ total on both dice

$X \backslash T$	2	3	4	5	6	7	8	9	10	11	12
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

Continuous joint distributions

- X takes values in $[a, b]$, Y takes values in $[c, d]$
- (X, Y) takes values in $[a, b] \times [c, d]$.
- Joint probability density function (pdf) $f(x, y)$

$f(x, y) \, dx \, dy$ is the probability of being in the small square.
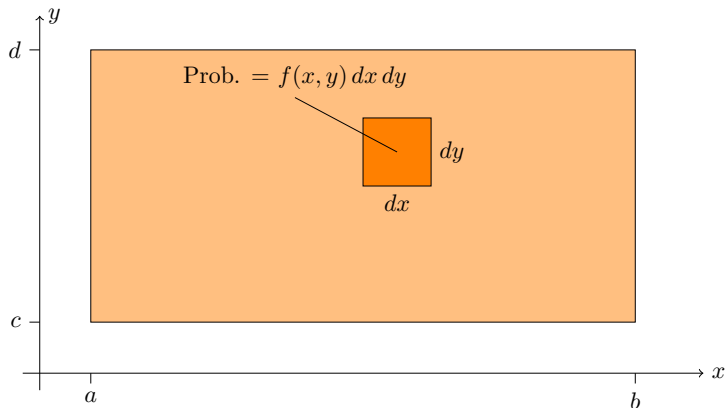
Properties of the joint pmf and pdf

Discrete case: probability mass function (pmf)

1. $0 \leq p(x_i, y_j) \leq 1$

2. Total probability is 1.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Continuous case: probability density function (pdf)

1. $0 \leq f(x, y)$

2. Total probability is 1.

$$\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = 1$$

Note: $f(x, y)$ can be greater than 1: it is a density *not* a probability.

Example: discrete events

Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.

Consider the event: $A = {}'Y - X \geq 2'$

Describe the event A and find its probability.

answer: We can describe A as a set of (X, Y) pairs:

$A = \{(1,3), (1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,5), (3,6), (4,6)\}.$

Or we can visualize it by shading the table:

$X\backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

$P(A) = $ sum of probabilities in shaded cells $= 10/36$.
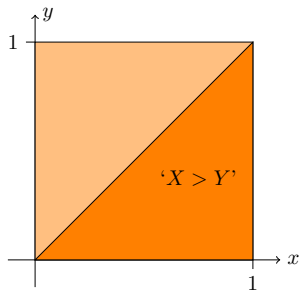
Example: continuous events

Suppose (X, Y) takes values in $[0, 1] \times [0, 1]$.

Uniform density $f(x, y) = 1$.

Visualize the event '$X > Y$' and find its probability.

answer:

The event takes up half the square. Since the density is uniform this is half the probability. That is, $P(X > Y) = 0.5$

Cumulative distribution function

$$F(x, y) = P(X \leq x, Y \leq y) = \int_c^y \int_a^x f(u, v) \, du \, dv.$$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y).$$

Properties

1. $F(x, y)$ is non-decreasing. That is, as x or y increases $F(x, y)$ increases or remains constant.

2. $F(x, y) = 0$ at the lower left of its range.
   If the lower left is $(-\infty, -\infty)$ then this means

   $$\lim_{(x,y) \to (-\infty, -\infty)} F(x, y) = 0.$$

3. $F(x, y) = 1$ at the upper right of its range.

Marginal pmf and pdf

Roll two dice: $X = \#$ on first die, $T = $ total on both dice.

The marginal pmf of X is found by summing the rows. The marginal pmf of T is found by summing the columns

$X \backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(t_j)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

For continuous distributions the marginal pdf $f_X(x)$ is found by integrating out the y. Likewise for $f_Y(y)$.

Board question

Suppose X and Y are random variables and

- (X, Y) takes values in $[0, 1] \times [0, 1]$.
- the pdf is $\dfrac{3}{2}(x^2 + y^2)$.

1. Show $f(x, y)$ is a valid pdf.
2. Visualize the event $A = $ '$X > 0.3$ and $Y > 0.5$'. Find its probability.
3. Find the cdf $F(x, y)$.
4. Find the marginal pdf $f_X(x)$. Use this to find $P(X < 0.5)$.
5. Use the cdf $F(x, y)$ to find the marginal cdf $F_X(x)$ and $P(X < 0.5)$.
6. See next slide

Board question continued

6. (New scenario) From the following table compute $F(3.5, 4)$.

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

answer: See next slide

Solution

answer: 1. Validity: Clearly $f(x, y)$ is positive. Next we must show that total probability $= 1$:

$$\int_0^1 \int_0^1 \frac{3}{2}(x^2 + y^2) \, dx \, dy = \int_0^1 \left[\frac{1}{2}x^3 + \frac{3}{2}xy^2\right]_0^1 \, dy = \int_0^1 \frac{1}{2} + \frac{3}{2}y^2 \, dy = 1.$$

2. Here's the visualization
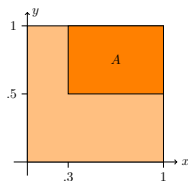
The pdf is not constant so we must compute an integral

$$P(A) = \int_{.3}^1 \int_{.5}^1 \frac{3}{2}(x^2 + y^2) \, dy \, dx = \int_{.3}^1 \left[\frac{3}{2}x^2y + \frac{1}{2}y^3\right]_{.5}^1 \, dx$$

(continued)

Solutions 2, 3, 4, 5

2. (continued) $\quad = \int_{.3}^{1} \frac{3x^2}{4} + \frac{7}{16}\,dx = \boxed{0.5495}$

3. $F(x, y) = \int_{0}^{y} \int_{0}^{x} \frac{3}{2}(u^2 + v^2)\,du\,dv = \boxed{\dfrac{x^3 y}{2} + \dfrac{xy^3}{2}}$.

4.

$$f_X(x) = \int_{0}^{1} \frac{3}{2}(x^2 + y^2)\,dy = \left[\frac{3}{2}x^2 y + \frac{y^3}{2}\right]_0^1 = \boxed{\frac{3}{2}x^2 + \frac{1}{2}}$$

$$P(X < .5) = \int_{0}^{.5} f_X(x)\,dx = \int_{0}^{.5} \frac{3}{2}x^2 + \frac{1}{2}\,dx = \left[\frac{1}{2}x^3 + \frac{1}{2}x\right]_0^{.5} = \boxed{\frac{5}{16}}.$$

5. To find the marginal cdf $F_X(x)$ we simply take y to be the top of the y-range and evalute F: $\quad F_X(x) = F(x, 1) = \frac{1}{2}(x^3 + x)$.

Therefore $\quad P(X < .5) = F(.5) = \frac{1}{2}(\frac{1}{8} + \frac{1}{2}) = \boxed{\frac{5}{16}}$.

6. On next slide

Solution 6

6. $F(3.5, 4) = P(X \leq 3.5, Y \leq 4)$.

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Add the probability in the shaded squares: $F(3.5, 4) = 12/36 = 1/3$.

Independence

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

Random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y).$$

Discrete random variables X and Y are independent if

$$p(x_i, y_j) = p_X(x_i)p_Y(y_j).$$

Continuous random variables X and Y are independent if

$$f(x, y) = f_X(x)f_Y(y).$$

Concept question: independence I

Roll two dice: $X =$ value on first, $Y =$ value on second

$X\backslash Y$	1	2	3	4	5	6	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(y_j)$	1/6	1/6	1/6	1/6	1/6	1/6	1

Are X and Y independent?   1. Yes   2. No

answer: 1. Yes. Every cell probability is the product of the marginal probabilities.

Concept question: independence II

Roll two dice: X = value on first, T = sum

$X \backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(y_j)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

Are X and Y independent?   1. Yes   2. No

answer: 2. No. The cells with probability zero are clearly not the product of the marginal probabilities.

Concept Question

Among the following pdf's which are independent? (Each of the ranges is a rectangle chosen so that $\int \int f(x,y)\,dx\,dy = 1$.)

(i) $f(x,y) = 4x^2y^3$.

(ii) $f(x,y) = \frac{1}{2}(x^3y + xy^3)$.

(iii) $f(x,y) = 6e^{-3x-2y}$

Put a 1 for independent and a 0 for not-independent.

(a) 111   **(b)** 110   **(c)** 101   **(d)** 100

(e) 011   **(f)** 010   **(g)** 001   **(h)** 000

answer: *(c). Explanation on next slide.*

Solution

(i) $\boxed{\text{Independent.}}$ The variables can be separated: the marginal densities are $f_X(x) = ax^2$ and $f_Y(y) = by^3$ for some constants a and b with $ab = 4$.

(ii) $\boxed{\text{Not independent.}}$ X and Y are not independent because there is no way to factor $f(x, y)$ into a product $f_X(x)f_Y(y)$.

(iii) $\boxed{\text{Independent.}}$ The variables can be separated: the marginal densities are $f_X(x) = ae^{-3x}$ and $f_Y(y) = be^{-2y}$ for some constants a and b with $ab = 6$.

Covariance

Measures the degree to which two random variables vary together, e.g. height and weight of people.

X, Y random variables with means μ_X and μ_Y

$$\text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)).$$

Properties of covariance

Properties
1. $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$ for constants a, b, c, d.
2. $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$.
3. $\text{Cov}(X, X) = \text{Var}(X)$
4. $\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y$.
5. If X and Y are independent then $\text{Cov}(X, Y) = 0$.
6. **Warning:** The converse is not true, if covariance is 0 the variables might not be independent.

Concept question

Suppose we have the following joint probability table.

$Y \backslash X$	-1	0	1	$p(y_j)$
0	0	1/2	0	1/2
1	1/4	0	1/4	1/2
$p(x_i)$	1/4	1/2	1/4	1

At your table work out the covariance $\text{Cov}(X, Y)$.

Because the covariance is 0 we know that X and Y are independent

1. True   2. False

Key point: covariance measures the linear relationship between X and Y. It can completely miss a quadratic or higher order relationship.

Board question: computing covariance

Flip a fair coin 12 times.

Let $X =$ number of heads in the first 7 flips

Let $Y =$ number of heads on the last 7 flips.

Compute $\text{Cov}(X, Y)$,

Solution

Use the properties of covariance.

X_i = the number of heads on the ith flip. (So $X_i \sim$ Bernoulli(.5).)

$$X = X_1 + X_2 + \ldots + X_7 \quad \text{and} \quad Y = X_6 + X_7 + \ldots + X_{12}.$$

We know $\text{Var}(X_i) = 1/4$. Therefore using Property 2 (linearity) of covariance

$$
\begin{aligned}
\text{Cov}(X, Y) &= \text{Cov}(X_1 + X_2 + \ldots + X_7, X_6 + X_7 + \ldots + X_{12}) \\
&= \text{Cov}(X_1, X_6) + \text{Cov}(X_1, X_7) + \text{Cov}(X_1, X_8) + \ldots + \text{Cov}(X_7, X_{12})
\end{aligned}
$$

Since the different tosses are independent we know

$$\text{Cov}(X_1, X_6) = 0, \ \text{Cov}(X_1, X_7) = 0, \ \text{Cov}(X_1, X_8) = 0, \text{ etc.}$$

Looking at the expression for $\text{Cov}(X, Y)$ there are only two non-zero terms

$$\text{Cov}(X, Y) = \text{Cov}(X_6, X_6) + \text{Cov}(X_7, X_7) = \text{Var}(X_6) + \text{Var}(X_7) = \boxed{\frac{1}{2}}.$$

Correlation

Like covariance, but removes scale.

The *correlation coefficient* between X and Y is defined by

$$\text{Cor}(X, Y) = \rho = \frac{\text{Cov}(X, Y)}{\sigma_X \, \sigma_Y}.$$

Properties:

1. ρ is the covariance of the standardized versions of X and Y.

2. ρ is dimensionless (it's a ratio).

3. $-1 \leq \rho \leq 1$. $\rho = 1$ if and only if $Y = aX + b$ with $a > 0$ and $\rho = -1$ if and only if $Y = aX + b$ with $a < 0$.

Real-life correlations

- Over time, amount of Ice cream consumption is correlated with number of pool drownings.

- In 1685 (and today) being a student is the most dangerous profession.

- In 90% of bar fights ending in a death the person who started the fight died.

- Hormone replacement therapy (HRT) is correlated with a lower rate of coronary heart disease (CHD).

Discussion is on the next slides.

Real-life correlations discussion

- Ice cream does not cause drownings. Both are correlated with summer weather.

- In a study in 1685 of the ages and professions of deceased men, it was found that the profession with the lowest average age of death was "student." But, being a student does not cause you to die at an early age. Being a student means you *are* young. This is what makes the average of those that die so low.

- A study of fights in bars in which someone was killed found that, in 90% of the cases, the person who started the fight was the one who died.

  Of course, it's the person who survived telling the story.

Continued on next slide

(continued)

- In a widely studied example, numerous epidemiological studies showed that women who were taking combined hormone replacement therapy (HRT) also had a lower-than-average incidence of coronary heart disease (CHD), leading doctors to propose that HRT was protective against CHD. But randomized controlled trials showed that HRT caused a small but statistically significant increase in risk of CHD. Re-analysis of the data from the epidemiological studies showed that women undertaking HRT were more likely to be from higher socio-economic groups (ABC1), with better-than-average diet and exercise regimens. The use of HRT and decreased incidence of coronary heart disease were coincident effects of a common cause (i.e. the benefits associated with a higher socioeconomic status), rather than cause and effect, as had been supposed.

Correlation is not causation

Edward Tufte: "Empirically observed covariation is a necessary but not sufficient condition for causality."

Overlapping sums of uniform random variables

We made two random variables X and Y from overlapping sums of uniform random variables

For example:

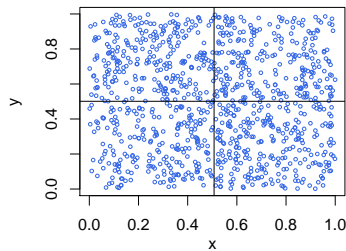$$X = X_1 + X_2 + X_3 + X_4 + X_5$$
$$Y = X_3 + X_4 + X_5 + X_6 + X_7$$

These are sums of 5 of the X_i with 3 in common.
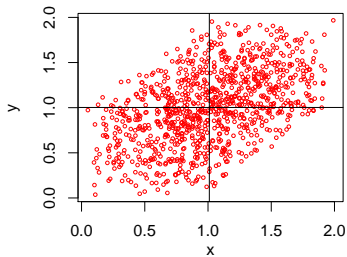
If we sum r of the X_i with s in common we name it (r, s).

Below are a series of scatterplots produced using R.

Scatter plots
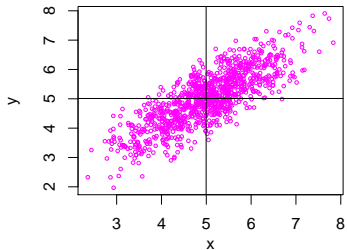

(1, 0) cor=0.00, sample_cor=−0.07

(2, 1) cor=0.50, sample_cor=0.48

(5, 1) cor=0.20, sample_cor=0.21
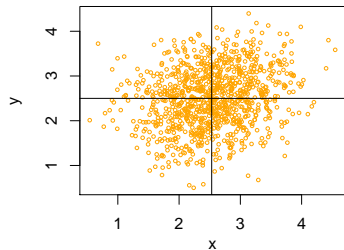
(10, 8) cor=0.80, sample_cor=0.81

Concept question

Toss a fair coin $2n + 1$ times. Let X be the number of heads on the first $n + 1$ tosses and Y the number on the last $n + 1$ tosses.

If $n = 1000$ then $\text{Cov}(X, Y)$ is:

(a) 0   (b) 1/4   (c) 1/2   (d) 1

(e) More than 1   (f) tiny but not 0

answer: 2. 1/4. This is computed in the answer to the next table question.

Board question

Toss a fair coin $2n + 1$ times. Let X be the number of heads on the first $n + 1$ tosses and Y the number on the last $n + 1$ tosses.

Compute $\text{Cov}(X, Y)$ and $\text{Cor}(X, Y)$.

As usual let X_i = the number of heads on the ith flip, i.e. 0 or 1. Then

$$X = \sum_1^{n+1} X_i, \qquad Y = \sum_{n+1}^{2n+1} X_i$$

X is the sum of $n + 1$ independent Bernoulli(1/2) random variables, so

$$\mu_X = E(X) = \frac{n + 1}{2}, \quad \text{and} \quad \text{Var}(X) = \frac{n + 1}{4}.$$

Likewise, $\mu_Y = E(Y) = \dfrac{n + 1}{2}$, and $\text{Var}(Y) = \dfrac{n + 1}{4}$.

Continued on next slide.

Solution continued

Now,

$$\text{Cov}(X, Y) = \text{Cov}\left(\sum_1^{n+1} X_i \sum_{n+1}^{2n+1} X_j\right) = \sum_{i=1}^{n+1} \sum_{j=n+1}^{2n+1} \text{Cov}(X_i X_j).$$

Because the X_i are independent the only non-zero term in the above sum is $\text{Cov}(X_{n+1} X_{n+1}) = \text{Var}(X_{n+1}) = \dfrac{1}{4}$ Therefore,

$$\text{Cov}(X, Y) = \frac{1}{4}.$$

We get the correlation by dividing by the standard deviations.

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{1/4}{(n+1)/4} = \frac{1}{n+1}.$$

This makes sense: as n increases the correlation should decrease since the contribution of the one flip they have in common becomes less important.

18.05 Introduction to Probability and Statistics
Spring 2014