Discrete Random Variables; Expectation
18.05 Spring 2014

FIG. 7.   FIG. 8.   FIG. 9.
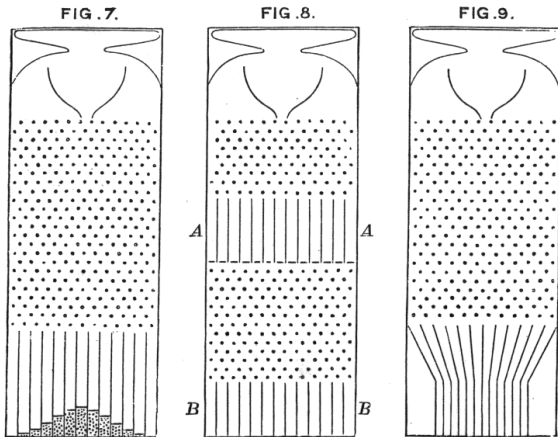
https://en.wikipedia.org/wiki/Bean_machine#/media/File:
Quincunx_(Galton_Box)_-_Galton_1889_diagram.png
http://www.youtube.com/watch?v=9xUBhhM4vbM

Reading Review

Random variable X assigns a number to each outcome:

$$X : \Omega \to \mathbf{R}$$

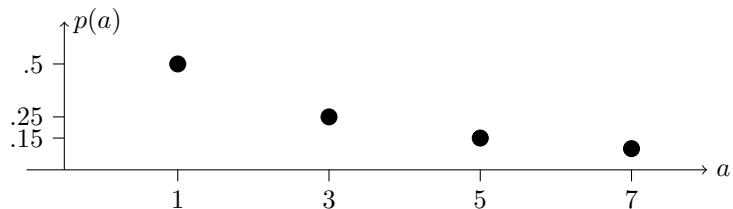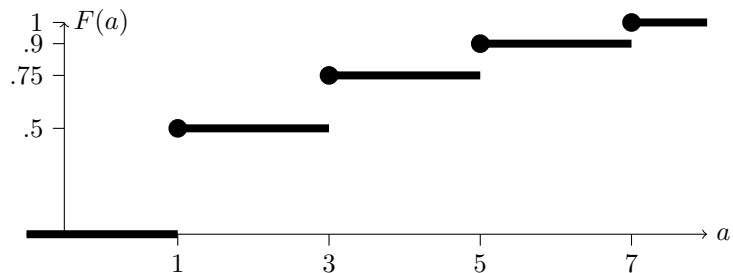"$X = a$" denotes the event $\{\omega \mid X(\omega) = a\}$.

Probability mass function (pmf) of X is given by

$$p(a) = P(X = a).$$

Cumulative distribution function (cdf) of X is given by

$$F(a) = P(X \leq a).$$

CDF and PMF

Concept Question: cdf and pmf

X a random variable.

values of X:	1	3	5	7
cdf $F(a)$:	0.5	0.75	0.9	1

1. What is $P(X \leq 3)$?

(a) 0.15   (b) 0.25   (c) 0.5   (d) 0.75

2. What is $P(X = 3)$

(a) 0.15   (b) 0.25   (c) 0.5   (d) 0.75

Deluge of discrete distributions

Bernoulli(p) =   1 (success) with probability p,
                   0 (failure) with probability $1 - p$.

In more neutral language:
Bernoulli(p) =   1 (heads) with probability p,
                   0 (tails) with probability $1 - p$.

Binomial(n,p) = # of successes in n independent
                        Bernoulli(p) trials.

Geometric(p) = # of tails before first heads in a
                        sequence of indep. Bernoulli(p) trials.
(Neutral language avoids confusing whether we want the number of
successes before the first failure or vice versa.)

Concept Question

1. Let $X \sim \text{binom}(n, p)$ and $Y \sim \text{binom}(m, p)$ be independent. Then $X + Y$ follows:

(a) $\text{binom}(n + m, p)$ (b) $\text{binom}(nm, p)$

(c) $\text{binom}(n + m, 2p)$ (d) other

2. Let $X \sim \text{binom}(n, p)$ and $Z \sim \text{binom}(n, q)$ be independent. Then $X + Z$ follows:

(a) $\text{binom}(n, p + q)$ (b) $\text{binom}(n, pq)$

(c) $\text{binom}(2n, p + q)$ (d) other

Board Question: Find the pmf

$X = \#$ of successes before the *second* failure of a sequence of independent Bernoulli(p) trials.

Describe the pmf of X.

Hint: this requires some counting.

Dice simulation: geometric(1/4)

Roll the 4-sided die repeatedly until you roll a 1.
Click in $X = \#$ of rolls *BEFORE* the 1.
(If X is 9 or more click 9.)

Example: If you roll (3, 4, 2, 3, 1) then click in 4.
Example: If you roll (1) then click 0.

Fiction

Gambler's fallacy: [roulette] if black comes up several times in a row then the next spin is more likely to be red.

Hot hand: NBA players get 'hot'.

Fact

P(red) remains the same.

The roulette wheel has no memory. (Monte Carlo, 1913).

The data show that player who has made 5 shots in a row is no more likely than usual to make the next shot.
(Currently, there seems to be some disagreement about this.)

Amnesia

Show that *Geometric*(p) is memoryless, i.e.

$$P(X = n + k \mid X \geq n) = P(X = k)$$

Explain why we call this memoryless.

Expected Value

X is a random variable takes values $x_1, x_2, \ldots, x_n$:

The **expected value** of X is defined by

$$E(X) = p(x_1)x_1 + p(x_2)x_2 + \ldots + p(x_n)x_n = \sum_{i=1}^{n} p(x_i)\, x_i$$

- It is a weighted average.
- It is a measure of central tendency.

Properties of $E(X)$

- $E(X + Y) = E(X) + E(Y)$ (linearity I)
- $E(aX + b) = aE(X) + b$ (linearity II)
- $E(h(X)) = \sum_{i} h(x_i)\, p(x_i)$

Examples

Example 1. Find $E(X)$

1.	X:	3	4	5	6
2.	pmf:	1/4	1/2	1/8	1/8

3. $E(X) = 3/4 + 4/2 + 5/8 + 6/8 = 33/8$

Example 2. Suppose $X \sim$ Bernoulli(p). Find $E(X)$.

1.	X:	0	1
2.	pmf:	$1 - p$	p

3. $E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$.

Example 3. Suppose $X \sim$ Binomial(12, .25). Find $E(X)$.
$X = X_1 + X_2 + \ldots + X_{12}$, where $X_i \sim$ Bernoulli(.25). Therefore

$$E(X) = E(X_1) + E(X_2) + \ldots E(X_{12}) = 12 \cdot (.25) = 3$$

In general if $X \sim$ Binomial(n, p) then $E(X) = np$.

Board Question: Interpreting Expectation

(a) Would you accept a gamble that offers a 10% chance to win $95 and a 90% chance of losing $5?

(b) Would you pay $5 to participate in a lottery that offers a 10% percent chance to win $100 and a 90% chance to win nothing?

• Find the expected value of your change in assets in each case?

Board Question

Suppose (hypothetically!) that everyone at your table got up, ran around the room, and sat back down randomly (i.e., all seating arrangements are equally likely).

What is the expected value of the number of people sitting in their original seat?

(We will explore this with simulations in Friday Studio.)

18.05 Introduction to Probability and Statistics
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