Bootstrapping

18.05 Spring 2014

Agenda

- Bootstrap terminology
- Bootstrap principle
- Empirical bootstrap
- Parametric bootstrap

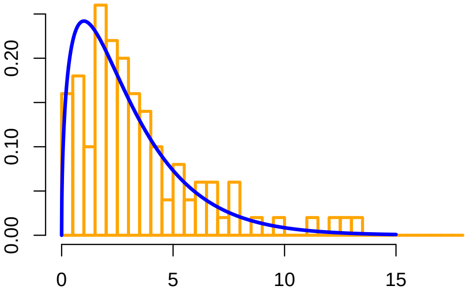
Empirical distribution of data

Data: x_1, x_2, ..., x_n (independent)

Example 1. Data: 1, 2, 2, 3, 8, 8, 8.

x^*	1	2	3	8
$p^*(x^*)$	$1/7$	$2/7$	$1/7$	$3/7$

Example 2.



The true and empirical distribution are approximately equal.

Resampling

- Sample (size 6): 1 2 1 5 1 12
- Resample (size m): Randomly choose m samples with replacement from the original sample.
- Resample probabilities = empirical distribution: $P(1) = 1/2$, $P(2) = 1/6$ etc.
- E.g. resample (size 10): 5 1 1 1 12 1 2 1 1 5
- A bootstrap (re)sample is always the same size as the original sample:
- Bootstrap sample (size 6): 5 1 1 1 12 1

Bootstrap principle for the mean

- Data $x_1, x_2, \ldots, x_n \sim F$ with true mean μ.
- $F^* =$ empirical distribution (resampling distribution).

•$x_1^*, x_2^*, \ldots, x_n^*$ resample same size data

Bootstrap Principle: (**really holds for any statistic**)

1. $F^* \approx F$ computed from the resample.
2. $\delta^* = \overline{x}^* - \overline{x} \approx \overline{x} - \mu \;=\;$ variation of $\overline{x}$

Critical values: $\delta^*_{1-\alpha/2} \le \overline{x}^* - \overline{x} \le \delta^*_{\alpha/2}$

then $\delta^*_{1-\alpha/2} \le \overline{x} - \mu \le \delta^*_{\alpha/2}$ so

$$\boxed{\overline{x} - \delta^*_{\alpha/2} \le \mu \le \overline{x} - \delta^*_{1-\alpha/2}}$$

Empirical bootstrap confidence intervals

Use the data to estimate the variation of estimates based on the data!

- Data: $x_1, \ldots, x_n$ drawn from a distribution F.
- Estimate a feature θ of F by a statistic $\hat{\theta}$.
- Generate many bootstrap samples $x_1^*, \ldots, x_n^*$.
- Compute the statistic θ^* for each bootstrap sample.
- Compute the bootstrap difference

$$\delta^* = \theta^* - \hat{\theta}.$$

- Use the quantiles of δ^* to approximate quantiles of

$$\delta = \hat{\theta} - \theta$$

- Set a confidence interval $[\hat{\theta} - \delta^*_{1-\alpha/2}, \hat{\theta} - \delta^*_{\alpha/2}]$
  (By $\delta_{\alpha/2}$ we mean the $\alpha/2$ **quantile**.)

Concept question

Consider finding bootstrap confidence intervals for

I. the mean **II.** the median **III.** 47th percentile.

Which is easiest to find?

A. I **B.** II **C.** III **D.** I and II

E. II and III **F.** I and III **G.** I and II and III

answer: **G.** The program is essentially the same for all three statistics. All that needs to change is the code for computing the specific statistic.

Board question

Data: 3 8 1 8 3 3

Bootstrap samples (each column is one bootstrap trial):

8	8	1	8	3	8	3	1
1	3	3	1	3	8	3	3
3	1	1	8	1	3	3	8
8	1	3	1	3	3	8	8
3	3	1	8	8	3	8	3
3	8	8	3	8	3	1	1

Compute a bootstrap 80% confidence interval for the mean.

Compute a bootstrap 80% confidence interval for the median.

Solution: mean

$\bar{x} = 4.33$

$\bar{x}^*$: `4.33, 4.00, 2.83, 4.83, 4.33, 4.67, 4.33, 4.00`

δ^*: `0.00, -0.33, -1.50, 0.50, 0.00, 0.33, 0.00, -0.33`

Sorted
δ^*: `-1.50, -0.33, -0.33, 0.00, 0.00, 0.00, 0.33, 0.50`

So, $\delta^*_{0.9} = -1.50$, $\delta^*_{0.1} = 0.37$.

(For $\delta^*_{0.1}$ we interpolated between the top two values –there are other reasonable choices. In R see the `quantile()` function.)

80% bootstrap CI for mean: $[\bar{x} - 0.37,\ \bar{x} + 1.50] = [3.97, 5.83]$

Solution: median

$x_{0.5} = \text{median}(x) \;=\; 3$

$x^*_{0.5}$: `3.0, 3.0, 2.0, 5.5, 3.0, 3.0, 3.0, 3.0`

δ^*: `0.0, 0.0, -1.0, 2.5, 0.0, 0.0, 0.0, 0.0`

Sorted
δ^*: `-1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 2.5`

So, $\delta^*_{0.9} = -1.0$, $\delta^*_{0.1} = 0.5$.

(For $\delta^*_{0.1}$ we interpolated between the top two values –there are other reasonable choices. In R see the `quantile()` function.)

80% bootstrap CI for median: $[\bar{x} - 0.5,\ \bar{x} + 1.0] = [2.5, 4.0]$

Empirical bootstrapping in R

```
x = c(30,37,36,43,42,43,43,46,41,42)   # original sample
n = length(x)        # sample size
xbar = mean(x)       # sample mean
nboot = 5000         # number of bootstrap samples to use

# Generate nboot empirical samples of size n
# and organize in a matrix
tmpdata = sample(x,n*nboot, replace=TRUE)
bootstrapsample = matrix(tmpdata, nrow=n, ncol=nboot)

# Compute bootstrap means xbar* and differences delta*
xbarstar = colMeans(bootstrapsample)
deltastar = xbarstar - xbar

# Find the .1 and .9 quantiles and make
# the bootstrap 80% confidence interval
d = quantile(deltastar, c(.1,.9))
ci = xbar - c(d[2], d[1])
```

Parametric bootstrapping

Use the estimated parameter to estimate the variation of estimates of the parameter!

- Data: $x_1, \ldots, x_n$ drawn from a parametric distribution $F(\theta)$.
- Estimate θ by a statistic $\hat{\theta}$.
- **Generate many bootstrap samples from $F(\hat{\theta})$.**
- Compute the statistic θ^* for each bootstrap sample.
- Compute the bootstrap difference

$$\delta^* = \theta^* - \hat{\theta}.$$

- Use the quantiles of δ^* to approximate quantiles of

$$\delta = \hat{\theta} - \theta$$

- Set a confidence interval $[\hat{\theta} - \delta^*_{1-\alpha/2}, \, \hat{\theta} - \delta^*_{\alpha/2}]$

Parametric sampling in R

```
# Data from binomial(15, θ) for an unknown θ
x = c(3, 5, 7, 9, 11, 13)
binomSize = 15     # known size of binomial
n = length(x)      # sample size
thetahat = mean(x)/binomSize     # MLE for θ
nboot = 5000       # number of bootstrap samples to use

# nboot parametric samples of size n; organize in a matrix
tmpdata = rbinom(n*nboot, binomSize, thetahat)
bootstrapsample = matrix(tmpdata, nrow=n, ncol=nboot)

# Compute bootstrap means thetahat* and differences delta*
thetahatstar = colMeans(bootstrapsample)/binomSize
deltastar = thetahatstar - thetahat

# Find quantiles and make the bootstrap confidence interval
d = quantile(deltastar, c(.1,.9))
ci = thetahat - c(d[2], d[1])
```

Board question

Data: 6 5 5 5 7 4 $\sim$ binomial(8,θ)

1. Estimate θ.

2. Write out the R code to generate data of 100 parametric bootstrap samples and compute an 80% confidence interval for θ.

(Try this without looking at your notes. We'll show the previous slide at the end)

Preview of linear regression

- Fit lines or polynomials to bivariate data
- Model: $y = f(x) + E$
  $f(x)$ function, E random error.
- Example: $y = ax + b + E$
- Example: $y = ax^2 + bx + c + E$
- Example: $y = \mathrm{e}^{ax+b+E}$ (Compute with $\ln(y) = ax + b + E$.)