18.04 Practice problems exam 2, Spring 2018

Problem 1. Harmonic functions
(a) Show $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and find a harmonic conjugate.

(b) Find all harmonic functions u on the unit disk such that $u(1/2) = 2$ and $u(z) \geq 2$ for all z in the disk.

(c) The temperature of the boundary of the unit disk is maintained at $T = 1$ in the first quadrant, $T = 2$ in the second quadrant, $T = 3$ in the third quadrant and $T = 4$ in the fourth quadrant. What is the temperature at the center of the disk

(d) Show that if u and v are conjugate harmonic functions then uv is harmonic.

(e) Show that if u is harmonic then u_x is harmonic.

(f) Show that if u is harmonic and u^2 is harmonic the u is constant.

(We always assume harmonic functions are real valued.)

Problem 2.
Let $f(z) = \dfrac{1}{(z-1)(z-3)}$. Find Laurent series for f on each of the 3 annular regions centered at $z = 0$ where f is analytic.

Problem 3.
Find the first few terms of the Laurent series around 0 for the following.

(a) $f(z) = z^2 \cos(1/3z)$ for $0 < |z|$.

(b) $f(z) = \dfrac{1}{e^z - 1}$ for $0 < |z| < R$. What is R?

Problem 4.
What is the annulus of convergence for $\displaystyle\sum_{n=-\infty}^{\infty} \frac{z^n}{2^{|n|}}$.

Problem 5.
Find and classify the isolated singularities of each of the following. Compute the residue at each such singularity.

(a) $f_1(z) = \dfrac{z^3 + 1}{z^2(z+1)}$

(b) $f_2(z) = \dfrac{1}{e^z - 1}$

(c) $f_3(z) = \cos(1 - 1/z)$

Problem 6.
(a) Find a function f that has a pole of order 2 at $z = 1 + i$ and essential singularies at $z = 0$ and $z = 1$.

(b) Find a function f that has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$ and an essential singularity at $z = i$.

Problem 7.
True or false. If true give an argument. If false give a counterexample

(a) If f and g have a pole at z_0 then $f + g$ has a pole at z_0.

(b) If f and g have a pole at z_0 and both have nonzero residues the fg has a pole at z_0 with a nonzero residue.

(c) If f has an essential singularity at $z = 0$ and g has a pole of finite order at $z = 0$ the $f + g$ has an essential singularity at $z = 0$.

(d) If $f(z)$ has a pole of order m at $z = 0$ then $f(z^2)$ has a pole of order $2m$

Problem 8.
Find the Laurent series for each of the following.

(a) $1/\mathrm{e}^{(1-z)}$ for $1 < |z|$.

Problem 9.
Let $h(z) = \dfrac{1}{\sin(z)} - \dfrac{1}{z} + \dfrac{2z}{z^2 - \pi^2}$ in the disk $|z| < 2\pi$.

(a) Show that all the apparent singularities are removable.

(b) Find the first 4 terms of the Taylor series around $z = 0$.

Problem 10.
Find the residue at ∞ of each of the following.

(a) $f(z) = \mathrm{e}^z$

(b) $f(z) = \dfrac{z-1}{z+1}$.

Problem 11.
Use the following steps to sketch the stream lines for the flow with complex potential $\Phi(z) = z + \log(z - i) + \log(z + i)$

(i) Identify the components, i.e. sources, sinks, etc of the flow.

(ii) Find the stagnation points.

(iii) Sketch the flow near each of the sources.

(iv) Sketch the flow far from the sources.

(v) Tie the picture together.

Problem 12.
Compute the following definite integrals

(a) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}\, d\theta$. (*Solution:* $\pi\sqrt{2}$)

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\, dx$. (*Solution:* $-\pi/27$)

(c) p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{1+x^2}\, dx$.

(d) p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x+i}\, dx$.

(e) $I =$ p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{x\mathrm{e}^{2ix}}{x^2-1}\, dx$.