Answers to Problem Set Number 3 for 18.04. MIT (Fall 1999)

Rodolfo R. Rosales* Boris Schlittgen† Zhaohui Zhang‡

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*MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

†MIT, Department of Mathematics, room 2-490, Cambridge, MA 02139.

‡MIT, Department of Mathematics, room 2-229, Cambridge, MA 02139.

1 Problems from the book by Saff and Snider.

1.1 Problem 04 in section 3.2.

Let us write $z = x + iy$. Then $\text{Log}(e^z) = \text{Log}(e^x e^{iy}) = \ln(e^x) + \text{Arg}(e^x e^{iy}) = x + i(y + 2k_0\pi)$ where k_0 is an integer chosen such that $-\pi < y + 2k_0\pi \leq \pi$. Thus, we see that $\text{Log}\, e^z = z$ if and only if $k_0 = 0$, which happens if and only if $-\pi < y \leq \pi$.

1.2 Problem 07 in section 3.2.

Let us write z in polar form: $z = re^{i\theta}$. Then the polar form of the Cauchy-Riemann equations is

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Writing $\log z = \ln(r) + i(\theta + 2k\pi)$, we have $u = \ln(r)$ and $v = \theta + 2k\pi$. Hence,

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}, \quad \frac{\partial v}{\partial r} = 0 \quad \text{and} \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0\,.$$

So we see that the Cauchy-Riemann equations are indeed satisfied, and thus $\log(z)$ is analytic.

To find its derivative, consider approaching the limit radially, with angle $\theta = \theta_0$ fixed. Then

$$\frac{d}{dz}\log(z) = \frac{d}{d(e^{i\theta_0}r)}(\ln(r) + i(\theta + 2k\pi)) = \frac{1}{e^{i\theta_0}}\frac{d}{dr}(\ln(r) + i(\theta + 2k\pi)) = \frac{1}{e^{i\theta_0}}\frac{1}{r} = \frac{1}{z}\,.$$

1.3 Problem 16 in section 3.2.

Write $z = \rho e^{i\theta}$, where $-\pi < \theta \leq \pi$. Then $w = \text{Log(z)} = \ln\rho + i\theta$. So the level curves for the real part of $\text{Log}(z)$ are curves with ρ =constant, i.e.: circles centered at the origin. The level curves for the imaginary part of $\text{Log}(z)$ are curves with constant argument θ, i.e. rays starting from the origin. See figure 1.3.1.

To see that the level curves are orthogonal at each point, we compute

$$\nabla(\text{Re}(w)) \cdot \nabla(\text{Im}(w)) = (1/\rho)\mathbf{e}_\rho \cdot (1/\rho)\mathbf{e}_\theta = 0\,.$$

Alternatively: the level curves for the real an imaginary parts of $\text{Log}(z)$ are the same as the coordinate lines for the (orthogonal and curvilinear) polar coordinate system.

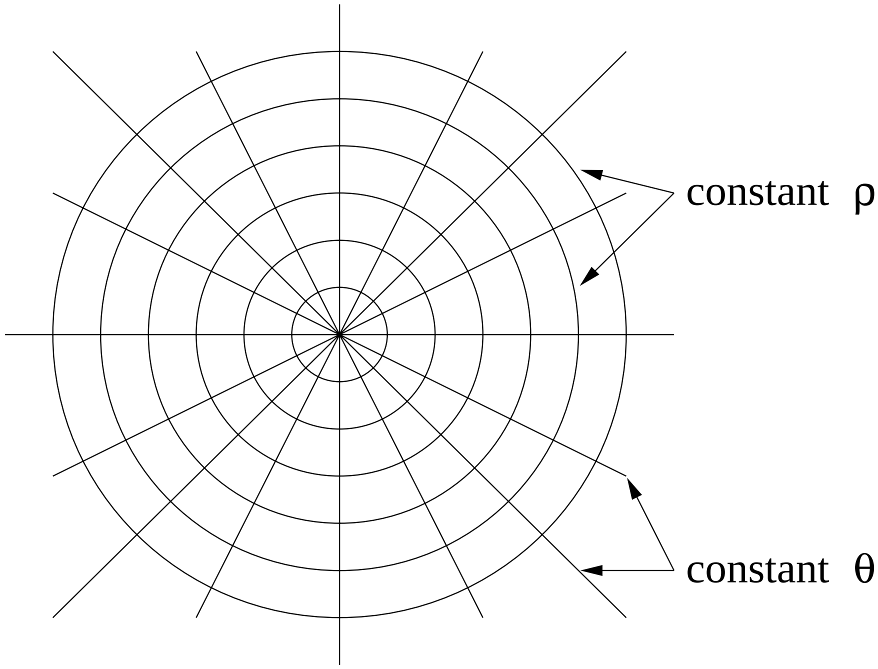

Figure 1.3.1: Level curves for the real and imaginary parts of Log(z).

1.4 Problem 09 in section 3.3.

Let $z = \cos(w) = \dfrac{1}{2}(e^{iw} + e^{-iw}) = \dfrac{1}{2}(\lambda + \lambda^{-1})$, where $\lambda = e^{iw}$. Multiplying both sides by 2λ and then using the quadratic formula to solve for λ, we find: $e^{iw} = \lambda = z + (z^2-1)^{1/2}$. Taking logs of both sides in this last equation and dividing by i, we find that

$$w = \cos^{-1}(z) = -i \log\left(z + (z^2-1)^{1/2}\right) . \tag{1.4.1}$$

Now we differentiate this expression to find

$$\frac{d}{dz}\cos^{-1}(z) = -i\frac{1}{z + (z^2-1)^{1/2}}\left(1 + \frac{z}{(z^2-1)^{1/2}}\right) = \frac{-i}{(z^2-1)^{1/2}} . \tag{1.4.2}$$

Remark 1.4.1 *We could now argue that, in the last expression in equation (1.4.2)*

$$i(z^2-1)^{1/2} = (1-z^2)^{1/2} , \tag{1.4.3}$$

and thus write

$$\frac{d}{dz}\cos^{-1}(z) = \frac{1}{(1-z^2)^{1/2}} . \tag{1.4.4}$$

We could equally make the argument that

$$i(z^2-1)^{1/2} = -(1-z^2)^{1/2}\,, \tag{1.4.5}$$

and so conclude that

$$\frac{d}{dz}\cos^{-1}(z) = \frac{-1}{(1-z^2)^{1/2}}\,, \tag{1.4.6}$$

which is formula (3.3.11) (page 92) in the book. However, now we seem to have arrived at two (apparently) different answers — equations (1.4.4) and (1.4.6) — for the same question! So **what is going on here?**

The answer to this conundrum lies in the multiple valued nature of the functions involved, and it also teaches us that we **have to be very careful** when dealing with multiple valued functions:

Both (1.4.3) and (1.4.5) are true only in the multiple valued sense, meaning that the set of values that the right hand sides can take is equal to the set of values that the left hand sides (respectively) can take. Since the values for square roots come in pairs with opposite signs, it is quite clear that these two equations are actually the same thing. *Equations (1.4.4) and (1.4.6) are valid in precisely the same way.* **However, equation (1.4.2) is valid in a somewhat stronger sense, as we explain next.**

Consider some arbitrary point $z_0 \neq \pm 1$ in the complex plane. In some neighborhood of it we can then define $(z^2-1)^{1/2}$ as a single valued function (i.e.: we pick a branch). Just so we do not get confused in the argument that follows, we will give **give a name** to this branch of $(z^2-1)^{1/2}$ — say: $G(z)$. Thus $G(z)$ is now some nice, single valued, analytic function defined in some neighborhood of z_0, which happens to have the property that $G^2 = z^2 - 1$.

Let us also choose a branch for the logarithm, defined in a neighborhood of $z_0 + G(z_0)$. Then we can write (using equation (1.4.1)):

$$w = \mathsf{arCoS}(z) = -i\,\mathsf{LoG}(z+G)\,, \tag{1.4.7}$$

where LoG is the name of the branch for the log we just selected and arCoS is the name for the branch of $\cos^{-1}$ that equation (1.4.7) defines. The important point that distinguishes equation (1.4.2) from equations (1.4.4) and (1.4.6) arises now, for we can substitute into it the various branches we have selected to obtain a true equation, valid in a single valued sense. That is:

$$\frac{d}{dz}\mathrm{arCoS}(z) = \frac{-i}{G(z)}\,. \tag{1.4.8}$$

This is easy to see, since all the operations in (1.4.2) are consistent with the definitions of the various branches above. But we **cannot do this with either (1.4.4) or (1.4.6)**, simply because these formulas involve yet another multiple valued function (namely: $(1-z^2)^{1/2}$) for which no branch has been selected.

So, **short and sweet:** once branches are defined for (1.4.1), equation (1.4.2) can be used to calculate the derivative, without any longer having to worry about possible multiple values.

Remark 1.4.2 *We can also write (this is equivalent to (1.4.1), as can be seen using (1.4.5))*

$$w = \cos^{-1}(z) = -i \log\left(z + i(1-z^2)^{1/2}\right) . \tag{1.4.9}$$

Then equation (1.4.6) has the same relationship to this formula that (1.4.2) has to (1.4.1). That is: once branches are defined for (1.4.9), equation (1.4.6) can be used to calculate the derivative, without any longer having to worry about possible multiple values.

1.5 Problem 12 in section 3.3.

Write $z = \tan w = -i(e^{iw} - e^{-iw})/(e^{iw} + e^{-iw})$ and solve for w. First multiply both sides of the equation by $ie^{iw}(e^{iw} + e^{-iw})$ to find $iz(e^{2iw} + 1) = e^{2iw} - 1$. Hence: $e^{2iw} = (1 + iz)/(1 - iz)$. Now take the log of both sides and divide by $2i$ to get:

$$w = \tan^{-1}(z) = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right) = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right) .$$

We differentiate now this expression using the chain rule:

$$\frac{d}{dz} \tan^{-1} z = \frac{i}{2}\left(\frac{i-z}{i+z}\right)\left(\frac{(i-z)-(i+z)(-1)}{(i-z)^2}\right) = \frac{1}{1+z^2} .$$

1.6 Problem 08 in section 4.1.

The contour Γ can be split naturally into two pieces, each one of which is easy to parametrize. Let us first parametrize these two curves separately and then patch them together in a second step. The first curve can be parametrized as: $z(t) = (-2+2i) + (1-2i)t$, for $0 \le t \le 1$. The second curve is described by: $z(t) = \exp(i\pi(1-t))$, for $0 \le t \le 1$. Patching these together, we have

$$\Gamma : z(t) = \begin{cases} (-2+2i) + (2-4i)t\,, & 0 \le t \le \frac{1}{2}\,; \\ e^{2i\pi(1-t)}\,, & \frac{1}{2} \le t \le 1\,. \end{cases}$$

We can now also parametrize the contour with the opposite orientation:

$$-\Gamma : z(t) = \begin{cases} e^{2i\pi(1+t)}\,, & -1 \le t \le -\frac{1}{2}\,; \\ (-2+2i) - (2-4i)t\,, & -\frac{1}{2} \le t \le 0\,. \end{cases}$$

1.7 Problem 14 in section 4.1.

Let

$$I \equiv \int_c^d \left| \frac{dz_2(s)}{ds} \right| = \int_c^d \left| \frac{dz_1(\phi(s))}{ds} \right| .$$

Now let $t = \phi(s)$ so that $dt = \dfrac{d\phi}{ds} ds = \left| \dfrac{d\phi}{ds} \right| ds$, since (by assumption) $d\phi/ds > 0$. Then:

$$I = \int_c^d \left| \frac{dz_1(t)}{dt} \frac{d\phi}{ds} \right| ds = \int_{t=\phi(c)}^{t=\phi(d)} \left| \frac{dz_1}{dt} \right| dt = \int_a^b \left| \frac{dz_1}{dt} \right| dt\,.$$

2 Other problems.

2.1 Problem 3.1 in 1999.

Statement: Consider the complex potential for a fluid given by $w = Az^3$, where $A > 0$ is a real number:

(i) Find the potential ϕ, the stream-function ψ and the velocity field (u, v).

(ii) Sketch the streamlines and the velocity field in the complex plane.

(iii) Can you use this to find an incompressible, irrotational flow in a wedge (for some angle)?

What is the angle of the wedge you can do with this solution?

Can you think of a way of getting solutions for other angles?

Solution: Let us set $A = 1$ for convenience (you should be able to see that none of the arguments below will change in any essential way if the size of A is changed).

- **i)** Let $= x + iy$. Then $w = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. Thus we can take:

$$\begin{aligned} \phi(x,y) &= \mathrm{Re}(w) = x^3 - 3xy^2 && \text{(potential)}, \\ \psi(x,y) &= \mathrm{Im}(w) = 3x^2y - y^3 && \text{(stream function)}, \\ \mathbf{u} &= \nabla\phi = 3(x^2 - y^2)\,\mathbf{i} - 6xy\,\mathbf{j} && (\mathbf{u} = (u, v) = \text{velocity field})\,. \end{aligned}$$

  Notice that $u - iv = 3z^2 =$ derivative of the complex potential $w = 3z^2$. This is always true; can you see why?

- **ii)** See figure 2.1.1.

- **iii)** We must impose the requirement that no fluid can escape across the boundary, that is (on the boundary): $\mathbf{u} \cdot \mathbf{n} = \nabla\phi \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal vector to the boundary. Since $\nabla\phi \cdot \nabla\psi = 0$ everywhere, the **boundary must be a stream-line:** $\psi =$ **constant.** Then $\nabla\psi$ is orthogonal to the boundary (on the boundary).

  Now clearly, $y = 0$ is one possible boundary, since $\psi(x, 0) = 0$. In order to obtain a wedge, we need another boundary of the form $y = \alpha x$, for some constant α. Since $\psi(x, \alpha x) = \alpha(3-\alpha^2)x^3$, for ψ to be constant along the line (i.e. independent of x), we need $\alpha = 0, \pm\sqrt{3}$. The case $\alpha = 0$ gives us the boundary we already have. The case $\alpha = \sqrt{3}$ gives us a boundary at an angle $\theta = \pi/3$ and the case $\alpha = -\sqrt{3}$ gives us a boundary at an angle $\theta = -\pi/3$. Thus we **can use the complex potential $w = z^3$ to describe an incompressible, irrotational, two dimensional flow in a wedge of angle $\pi/3$.** See figure 2.1.1.

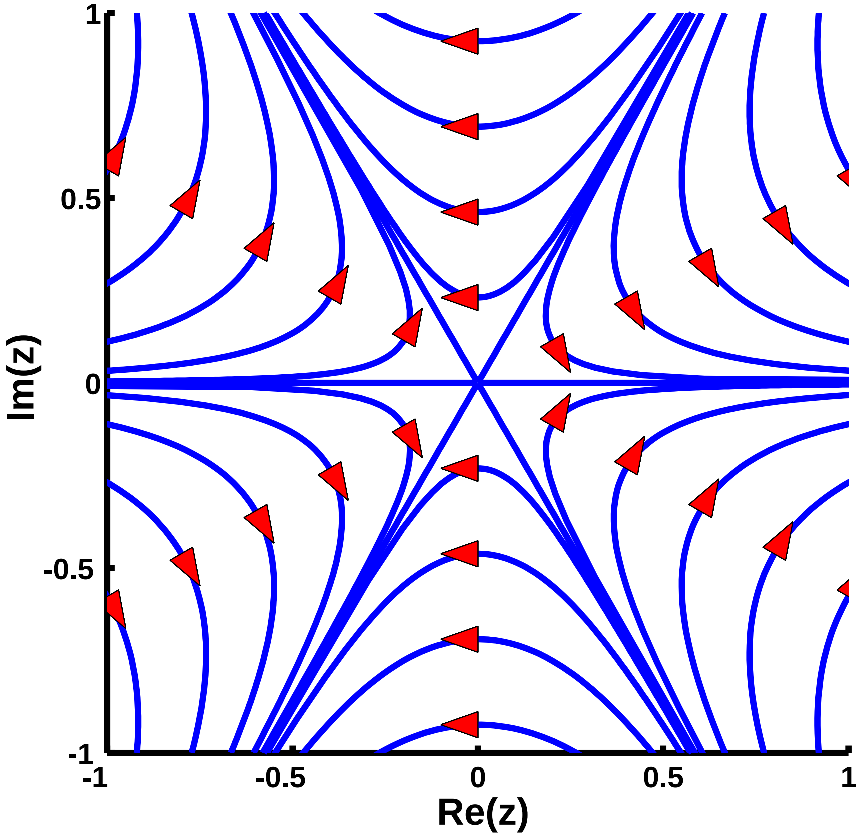

Figure 2.1.1: Stream lines for the flow given by $w = z^3$. The flow is along the stream lines, in the direction indicated by the arrows. The magnitude of the flow speed is $3r^2$, where $r = |z|$.

In order to obtain flow in wedges of different angles, consider $w = z^\beta = \rho^\beta e^{i\beta\theta}$. Then the stream function is given by $\psi = \text{Im}(w) = \rho^\beta \sin(\beta\theta)$ and two level curves where $\psi = \text{constant}$ are clearly given by $\theta = 0$ and $\theta = \pi/\beta$ (since ψ vanishes on these curves).

THE END.