PROFESSOR: Welcome back. So in this session, we're going to look at unit step and impulse responses. So in this question, we ask you to find the unit impulse response to these two equations, x dot plus 2x equals f of t, and 2 x dot dot plus 27 x dot-- oops, it should be a 7-- plus 7 x dot plus 3x equals f of t. In the second part, you're asked to find the unit step response for the first equation.

So here, the key points are really to remember what do we mean by unit impulse and unit step response. Which initial condition correspond to these responses? And what functions of f of t do you choose in each case? So why don't you pause the video and work through this problem? And I'll be right back.

Welcome back. So let's look at the equation A. So the unit impulse response is simply-- I'm going to write this down, unit impulse response-- is simply the solution to the following problem, to our differential equation, x dot plus 2x that we're given, with the forcing in a delta function of magnitude 1 with rest initial conditions, which means in this case, that x of 0 is equal to 0. x of 0 minus is equal to 0.

So how do we go about solving this equation? So here, first of all, let's look at what this means. This equation could be modeling, for example, the quantity of radioactive chemical in a tank. And delta(t) here means that we are giving a high disturbance on a very short time on this system. So for example, dumping a huge amount of chemical in that tank, and then, just letting the system evolve after dumping all that chemical in a tank.

So really, what we could see is that the delta can be identified to x dot, which is the highest derivative in the system. And that would then mean that we have a jump in the second highest derivative, which would be in x. So delta t introduces a jump in x. And what do I mean by that is that we had rest initial conditions initially. Now, we have to go into a jump where the quantity in the tank was raised to 1, which is basically the magnitude on the right-hand side and the fact that we have no coefficients for the highest derivative in the system.

So basically, we can solve this equation by solving the equivalent problem x dot plus 2x equals to 0, which is just the homogeneous part, with our new initial conditions, x of 0 minus equals to 1. And so here, clearly from before, we saw that the solution would be a decaying exponential. We would have a constant of integration here, which would be just 1 given our new initial condition.

So this would be valid for t larger than 0. And given that we started from rest initial conditions prior to this perturbance introduced by the delta, we could also write this with a u of t step function if we were to consider t in R. So if you want to look at how the solution looks like, we would have basically rest initial conditions, and then, a jump introduced by our forcing, and after that, just a decaying exponential, and a jump here to a new initial condition of 1.

So let's move to the second problem. Now, we're dealing with a second-order differential equation. 2 x dot dot plus 7 x dot plus 3x equals f of t. And we are asked again to seek the unit impulse response to this problem. So the unit impulse response is simply the solution of 2 x dot dot plus 7 x dot plus 3x equals-- Again, the unit impulse means that we're just kicking the system with an impulse or delta function. And we're starting from rest initial conditions.

So here, let's just look at what could this be modeling. So this could be just basically Newton's Second Law with acceleration and forces applied to the system. For example, it could be a mass hanged on a spring with certain damping due to this term. And this delta here would have basically the units corresponding to a force that would be felt by the highest order of term in the integral which could correspond to the acceleration multiplied, for example, by a mass of 2.

So really, the delta can be identified to the highest-order derivative, which means that we would have a discontinuity from 0 to 1 on the derivative one degree lower, so in x dot. And so the delta introduces the discontinuity on the second highest derivative. And what I mean by that is that we have a jump from our original initial conditions of x dot 0 minus equals to 0 to new initial conditions, x dot of 0 minus equals one half.

Where did this one half come from? It came from the fact that I have a factor of 2 in front of the highest-order derivative. So if I'm identifying this with delta, then integration of this term would give me an x dot that corresponds to the jump from 0 to 1 over 2, so one half. That's where it comes from. And each time that you would have a coefficient in front of your highest order derivative associated with the delta on the right, you would have a jump of 1 over that coefficient.

So from this point, what are we solving? What do we need to solve to get the unit impulse response? It's equivalent to solving our new system where we can then get rid of the delta with our new initial conditions, dot equals one half. And of course, here, I did mentioned by the initial condition on x 0 minus 1, because we need two initial conditions for the second order derivative, would still be 0, as the discontinuity would not be felt by the function x itself.

So we can go ahead and solve this problem. So we use the characteristic polynomial as we did before. This characteristic polynomial would have a discriminant of 25, which gives us simple roots that we can compute, so minus 7 plus or minus the square root of the discriminant over 4. So we have two roots, minus 7 minus 5 over 4, minus 12 over 4, which is minus 3; minus 7 plus 5 over 4, which just gives us minus one half.

So we can write down the solution to this problem as c_1 exponential the root that we just found, minus 3t, plus c_2 exponential the other root, minus one half of t. Now, to get c_1, c_2, we need to take into account the initial conditions. So the first one tells us that x at 0 is 0. So that will give us c_1 plus c_2 equals to 0, so basically, c_2 equals minus c_1. And the second initial condition tells us that the derivative is 0. So that gives us minus 3c_1 minus one half c_2 equals to 0.

Here, we have that c_2 is equal to minus c_1. So we can just factor out everything. And we end up with 3 minus one half-- whoops, sorry. Here it should be one half, our new modified initial condition, equals to one half. And here, you would have 5/2 equal to 1/2. So c_2 equals 1/5 equals minus c_1. And so plugging in c_1 and c_2 in this formula would give us the general solution with c_2 exponential minus one half t minus exponential minus 3t.

And all of this, remember, we're solving for t larger than 0 in our modified system with our new initial conditions. So this is for t larger than 0. And again, here, if we wanted to just write it for t in R, then we could just add the step function u of t that would just signify that we took rest initial conditions before x equals 0 minus.

So that ends the first part. So now quickly, for the second part, we're asked to find the unit step response to the first system. So the unit step response is just the solution to our original ODE problem, x dot plus 2x. But now, f of t is the step function, hence, the step here, with still rest initial conditions.

So really, the step function is just 0 everywhere before 0 minus. And it takes the value of 1 after. And so basically, we can just solve again for t positive, the modified differential equation with u taking just the value of 1. And here, we just get similar roots. So it would be a minus 2t. But then, we would have a constant of integration to worry about. And a new particular solution, a lucky guess would just be a constant. And that would give us the one half from 1 over 2.

So then, we just need to seek the c_1 that would give us x of t equals to 0 at 0. Because in this case, we don't need to modify the initial conditions. They still remain the same, rest initial conditions. And so we would get here just c_1 equals minus one half. So we have 1 minus exponential of minus 2t.

So just here, something I forgot to mention, what could this be modeling? So this could be modeling, if I go back to my analogy of the radioactive chemical in the tank, this would be telling me that after a certain time, I start inputting at a steady rate at a constant rate of u of t, of one per unit of time, the amount in the tank and then looking at how the system evolves to that. So there's no abrupt change that introduces a discontinuity.

So just to conclude, I just want to sketch the solution just so that you see the difference between the two. And here, what we have, again, I could have introduced my u of t here if I want t in R. So here, we have again a solution that is 0 before. There is no discontinuity. It's still 0 at 0 minus. And then, we have basically a solution that is growing until it reaches an asymptote of one half when this exponential goes to 0. And so you can see that it's a smooth transition because I just started inputting the amount of the chemical in the tank in a non-abrupt way.

So this concludes this session. And the key here was to really remember what do we mean by the unit step response. What type of f of t are we talking about? What initial conditions do we need? And same thing for the unit impulse response.