PROFESSOR: Hi, everyone. Welcome back. So today we're going to take a look at solving differential equations using the Laplace transforms, and the problem we're going to take a look at is a simple ODE, x-dot plus 2x equals 3 delta of t plus 5, as a forcing on the right hand side. We're going to look at having rest initial conditions, x of 0 minus is equal to 0, and we're asked to use Laplace transforms to solve this initial value problem. For part B, we're asked to have the initial value problem without any delta function forcing on the right-hand side to give an equivalent initial value problem without a delta function forcing on the right-hand side, but yields the same solution as in part A.

And then in question two, we're asked to solve the second-order differential equation, x-dot-dot plus 9x equals u of t with rest initial conditions, so x of 0 minus is 0, x-dot of 0 minus is also 0. So I'll let you work on this problem, and I'll be back in a moment. Hi, everyone. Welcome back.

OK, so for part A, the first step is to Laplace transform both sides of the equation. So we take the Laplace transform of x-dot plus 2x, and that's going to be equal to the Laplace transform of 3 delta of t plus 5. And we can use the distribution properties of the Laplace transform, so this is going to be Laplace transform of x-dot plus 2 times the Laplace transform of x. On the right-hand side, we have 3 times the Laplace transform of the delta function plus 5 times the Laplace transform of 1.

Now, we can replace the Laplace transform of x-dot if we use the identity with s times the Laplace transform of x minus x at 0 minus. And we're told that x of 0 minus in this case is just 0, so this term's going to vanish. And just for brevity, I'm going to write X of s as the Laplace transform of x. So we now have s times X of s plus 2 X of s equals, and on the right-hand side, we have three times the Laplace transform of the delta function. Laplace transform of the delta function is just 1, so we have 3 plus 5, and the Laplace transform of 1 is just 1 over s.

So I can now factor the left-hand side, and I get X of s times s plus 2 equals 3 plus 5 over s. And note how, when we have X of s multiplied by some polynomial in s, this is always going to be the characteristic polynomial. So if we look back, s plus 2 is the characteristic polynomial of x-dot plus 2x. So this yields 3 divided by s plus 2, on the right-hand side, plus 5 divided by s times s plus 2.

And for the second piece, we can use partial fractions to decompose it into a term times s and a term times s plus 2. And when we do that, we end up getting 5/2 1 over s minus 1 over s plus 2. So I can combine the 3 divided by s plus 2 with the minus 5/2 divided by s plus 2 into one term. So this gives you 1/2 1 over s plus 2, and we also have 5/2 1 over s.

And now we just take the inverse Laplace transform of both sides. So on the left-hand side, we recover x of t, so we get 1/2 e to the minus 2t plus 5/2. The inverse Laplace transform of 1/s is just 1. So we end up with x of t is 1/2 e to minus 2t plus 5/2, and this solution is valid for t bigger than 0. Sometimes, people write it as this quantity multiplied by step function.

And x of t is also 0 for t less than 0, for example. And it's just useful to quickly sketch what x of t looks like, so it's an exponential decay for t bigger than 0, and it's flat for t less than 0. So here's our x of t.

So for part B, now, we're asked to find a differential equation and new initial conditions that reproduce the solution offered t bigger than 0. So note how we'd be looking for a new solution, x of t, which would be an exponential decay. And essentially, we just grow, so we're looking at initial conditions, which start at 0.

If I were to write the original differential equation-- so this is the original differential equation from part A. Looks like this. And just quickly to note that x-dot near the origin is going to be approximately e times the delta function, which means in the original differential equation, x is going to have a jump of 3 about the origin.

So the new initial value problem-- well, we don't want the delta function on the right-hand side, so we're going to solve x-dot plus 2x is equal to 5. But what initial conditions do we need? Well, we need x of 0 minus to now be 3. So when we eliminate the 3 delta on the right-hand side, we have to introduce new initial conditions so that the solution agrees for t bigger than 0.

OK, so this concludes part one. For part two, we're asked to solve a new differential equation, x-dot-dot plus 9x equals u of t, and we're just going to follow the same procedure where we Laplace transform both sides. So Laplace transforming the left-hand side gives me x-dot-dot plus 9x equals the Laplace transform of u of t. And again, I can use the formula which relates derivatives of x back to the Laplace transform of x, and so in this case, the Laplace transform of x-dot-dot is going to be s squared times the Laplace transform of x.

And then I'm going to have plus a term which involves x of 0 minus and a term which involves x-dot of 0 minus. And if your initial conditions were not 0, you would have to keep these terms in. However in our case, these terms are both 0 because we deal with rest initial conditions, I'm just not going to write them. Plus 9 X of s equals-- the Laplace transform of u of t is 1/s.

So again, we have X of s s squared plus 9-- so note again how this is the same characteristic polynomial as in our differential equation-- is equal to 1/s. So X of s is 1/s s squared plus 9, which we can use partial fractions, now, to decompose it into A over s plus B*s plus C divided by s squared plus 9. And if I were to take a look at my notes, I have, in this case, A is 1/9, B is negative 1/9, and C is equal to 0, if you were to work out these coefficients.

So what this means is X of s is 1/9 1 over s minus 1/9 1 over s squared plus 9. And when we invert the Laplace transform, the inverse of 1/s is just 1, so x of t becomes 1 over 9. Sorry, this should be s up here. The inverse Laplace transform of s divided by s squared plus 9 is cosine 3, so we end up with negative 1/9 cosine of 3t, and again, this is a solution for t bigger than 0. So as soon as we turn on the input, the function x of t starts growing continuously from 1, and then achieves an oscillation with period 3.

So just to quickly recap, in this problem, we solved several ODEs, several initial value problem ODEs using Laplace transforms. Laplace transforms are particularly nice because they convert an ODE into an algebraic equation, which we can solve fairly easily. The drawback is we sometimes have to manipulate, using partial fractions, the right-hand side into functions that we know how to invert using the Laplace transform inverse. So I'll just conclude here, and I'll see you next time.