More Entries for the Laplace Table

In this note we will add some new entries to the table of Laplace transforms.

1. $\mathcal{L}(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2}$, with region of convergence $\mathrm{Re}(s) > 0$.

2. $\mathcal{L}(\sin(\omega t)) = \dfrac{\omega}{s^2+\omega^2}$, with region of convergence $\mathrm{Re}(s) > 0$.

Proof: We already know that $\mathcal{L}(e^{at}) = 1/(s-a)$. Using this and Euler's formula for the complex exponential, we obtain

$$\mathcal{L}(\cos(\omega t) + i\sin(\omega t)) = \mathcal{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{1}{s-i\omega}\cdot\frac{s+i\omega}{s+i\omega} = \frac{s+i\omega}{s^2+\omega^2}.$$

Taking the real and imaginary parts gives us the formulas.

$$\begin{aligned} \mathcal{L}(\cos(\omega t)) &= \mathrm{Re}\left(\mathcal{L}(e^{i\omega t})\right) = s/(s^2+\omega^2) \\ \mathcal{L}(\sin(\omega t)) &= \mathrm{Im}\left(\mathcal{L}(e^{i\omega t})\right) = \omega/(s^2+\omega^2) \end{aligned}$$

The region of convergence follow from the fact that $\cos(\omega t)$ and $\sin(\omega t)$ both have exponential order 0.

Another approach would have been to use integration by parts to compute the transforms directly from the Laplace integral.

3. For a positive integer n, $\mathcal{L}(t^n) = n!/s^{n+1}$. The region of convergence is $\mathrm{Re}(s) > 0$.

Proof: We start with $n = 1$.

$$\mathcal{L}(t) = \int_0^\infty te^{-st}\,dt$$

Using integration by parts:

$$\left.\begin{array}{ll} u = t & dv = e^{-st} \\ du = 1 & v = e^{-st}/(-s) \end{array}\right\} \mathcal{L}(t) = -\frac{te^{-st}}{s}\bigg]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt.$$

For $\mathrm{Re}(s) > 0$ the first term is 0 and the second term is $\frac{1}{s}\mathcal{L}(1) = 1/s^2$. Thus, $\mathcal{L}(t) = 1/s^2$.

Next let's do $n = 2$:

$$\mathcal{L}(t^2) = \int_0^\infty t^2e^{-st}\,dt$$

Again using integration by parts:

$$\left.\begin{array}{ll} u = t^2 & dv = e^{-st} \\ du = 2t & v = e^{-st}/(-s) \end{array}\right\} \mathcal{L}(t^2) = -\frac{t^2 e^{-st}}{s}\bigg]_0^\infty + \frac{1}{s}\int_0^\infty 2te^{-st}\,dt.$$

For $\mathrm{Re}(s) > 0$ the first term is 0 and the second term is $\frac{1}{s}\mathcal{L}(2t) = 2/s^3$. Thus, $\mathcal{L}(t^2) = 2/s^3$.

We can see the pattern: there is a reduction formula for

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\,dt.$$

Integration by parts:

$$\left.\begin{array}{ll} u = t^n & dv = e^{-st} \\ du = nt^{n-1} & v = e^{-st}/(-s) \end{array}\right\} \mathcal{L}(t^n) = -\frac{t^n e^{-st}}{s}\bigg]_0^\infty + \frac{1}{s}\int_0^\infty nt^{n-1}e^{-st}\,dt.$$

For $\mathrm{Re}(s) > 0$ the first term is 0 and the second term is $\frac{1}{s}\mathcal{L}(nt^{n-1})$. Thus, $\mathcal{L}(t^n) = \frac{n}{s}\mathcal{L}(t^{n-1})$.

Thus we have

$$\begin{array}{lllll} \mathcal{L}(t^3) & = & \frac{3}{s}\mathcal{L}(t^2) & = & \frac{3\cdot 2}{s^4} = \frac{3!}{s^4} \\ \mathcal{L}(t^4) & = & \frac{4}{s}\mathcal{L}(t^3) & = & \frac{4\cdot 3!}{s^5} = \frac{4!}{s^5} \\ \ldots & & & & \\ \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. & & & & \end{array}$$

4. (s-shift formula) If z is any complex number and $f(t)$ is any function then

$$\mathcal{L}(e^{zt}f(t)) = F(s-z).$$

As usual we write $F(s) = \mathcal{L}(f)(s)$. If the region of convergence for $\mathcal{L}(f)$ is $\mathrm{Re}(s) > a$ then the region of convergence for $\mathcal{L}(e^{zt}f(t))$ is $\mathrm{Re}(s) > Re(z) + a$.

Proof: We simply calculate

$$\begin{aligned} \mathcal{L}(e^{zt}f(t)) &= \textstyle\int_0^\infty e^{zt}f(t)e^{-st}\,dt \\ &= \textstyle\int_0^\infty f(t)e^{-(s-z)t}\,dt \\ &= F(s-z). \end{aligned}$$

Example. Find the Laplace transform of $e^{-t}\cos(3t)$.

Solution. We could do this by using Euler's formula to write

$$e^{-t}\cos(3t) = (1/2)\left(e^{(-1+3i)t} + e^{(-1-3i)t}\right)$$

but it's even easier to use the s-shift formula with $z = -1$, which gives

$$\mathcal{L}(e^{-t}f(t)) = F(s+1),$$

where here $f(t) = \cos(3t)$, so that $F(s) = s/(s^2+9)$. Shifting s by -1 according to the s-shift formula gives

$$\mathcal{L}(e^{-t}\cos(3t)) = F(s+1) = \frac{s+1}{(s+1)^2+9}.$$

We record two important cases of the s-shift formula:

4a) $\mathcal{L}(e^{zt}\cos(\omega t)) = \dfrac{s-z}{(s-z)^2+\omega^2}$

4b) $\mathcal{L}(e^{zt}\sin(\omega t)) = \dfrac{\omega}{(s-z)^2+\omega^2}$.

Consistency.
It is always useful to check for consistency among our various formulas:

1. We have $\mathcal{L}(1) = 1/s$, so the s-shift formula gives $\mathcal{L}(e^{zt}\cdot 1) = 1/(s-z)$. This matches our formula for $\mathcal{L}(e^{zt})$.

2. We have $\mathcal{L}(t^n) = n!/s^{n+1}$. If $n = 1$ we have $\mathcal{L}(t^0) = 0!/s = 1/s$. This matches our formula for $\mathcal{L}(1)$.