Fourier Series for Functions with Period 2L

Suppose that we have a periodic function $f(t)$ with arbitrary period $P = 2L$, generalizing the special case $P = 2\pi$ which we have already seen. Then a simple re-scaling of the interval $(-\pi, \pi)$ to $(-L, L)$ allows us to write down the general Fourier series and Fourier coefficent formulas:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + b_n \sin\left(n\frac{\pi}{L}t\right) \tag{1}$$

with Fourier coefficients given by the general Fourier coefficent formulas

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\frac{\pi}{L}t)\,dt, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\frac{\pi}{L}t)\,dt. \end{aligned} \tag{2}$$

Note: The number $L = \frac{P}{2}$ is called the **half-period**.

Example. Let $f(t)$ be the period 2 function, which is defined on the window $[-1, 1)$ by $f(t) = |t|$. Compute the Fourier series of $f(t)$.

The graph of $f(t)$ below shows why this function is called either a triangle wave or a continuous sawtooth function.

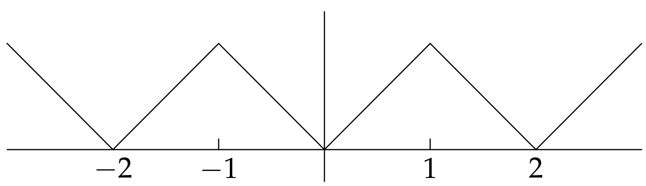

Figure 1: The period 2 triangle wave.

Solution. In this case the period is $P = 2$, so the half-period $L = 1$. This means $n\frac{\pi}{L} = n\pi$ and we compute the coefficients from the formulas in (2), using integration by parts, as follows.

For $n \neq 0$:

$$\begin{aligned} a_n &= \frac{1}{1}\int_{-1}^{1} |t|\cos(n\pi t)\,dt = 2\int_0^1 t\cos(n\pi t)\,dt \\ &= 2\left(\frac{t\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{n^2\pi^2}\right|_0^1 = \frac{2}{n^2\pi^2}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for n even} \end{cases} \end{aligned}$$

and for $n = 0$:

$$a_0 = \frac{1}{1}\int_{-1}^{1} |t|\,dt = 2\int_0^1 t\,dt = 1$$

Since $f(t)$ is an even function and $\sin(n\pi t)$ is odd, the sine coefficients $b_n = 0$. (We will justify this carefully in the next session. For now you can compute the integrals for b_n as an exercise and verify it in this case.)

Thus, the Fourier series for $f(t)$ is

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \cdots\right) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.$$