Fourier Series: Definitions and Coefficients

We will first state Fourier's theorem for periodic functions with period $P = 2\pi$. In words, the theorem says that a function with period 2π can be written as a sum of cosines and sines which all have period 2π.

Theorem (Fourier)
Suppose $f(t)$ has period 2π then we have

$$
\begin{aligned}
f(t) \quad \sim \quad & \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \ldots \\
& + b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \ldots \\
= \quad & \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt),
\end{aligned} \tag{1}
$$

where the coefficients $a_0, a_1, \ldots$ and $b_1, b_2 \ldots$ are computed by

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt
\end{aligned} \tag{2}
$$

Some comments are in order.

1. As we saw in the quiz above, each of the functions $\cos(t), \cos(2t), \cos(3t), \ldots$ all have 2π as a period. The same is clearly true for $\sin(t), \sin(2t), \sin(3t), \ldots$.

2. The series on the right-hand side (1) is called a **Fourier series**; and the coefficients , $a_0, a_1, \ldots b_1, b_2, \ldots$ in (2) are called the **Fourier coefficients** of $f(t)$.

3. The letter a is used in $a_0/2$ because we can think of it as the coefficient of $\cos(0 \cdot t) = 1$. We don't need a b_0 term because $\sin(0 \cdot t) = 0$. The term constant term $\frac{a_0}{2}$ is written in this way to make the formula for a_0 look just like those of the other cosine coefficients a_n. (We will see why we need the factor of $\frac{1}{2}$ in a later note when we prove that these formulas really do give the coefficients.)

4. In (1) we used the symbol $\sim$ instead of an equal sign because the two sides of (1) might differ at those values of t where $f(t)$ is discontinuous. For us, this is a minor point and we will allow ourselves to use an equal sign from now on.

5. There is some terminology coming from acoustics and music: the $n = 1$ frequency is called the **fundamental**, and the frequencies $n \geq 2$ are called the **higher harmonics** (or overtones). We will explore the connection between Fourier series and sound in a later session.

Fourier series are a wonderful tool for breaking a periodic function, however complicated, into simple pieces. The superposition principle will then allow us to solve DE's with arbitrary periodic input in Fourier series form.

In later notes we will extend Fourier's theorem to functions of other periods. The extension is straighforward, but requires more notation, so we will wait until you have gained some experience with Fourier series.

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