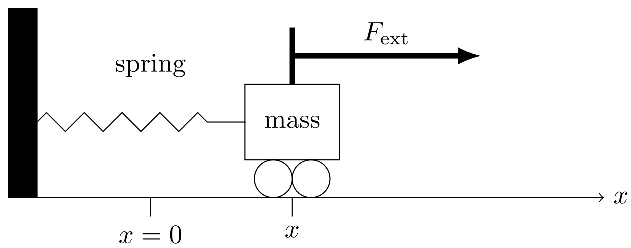
Second Order Physical Systems

1. Second Order Physical Systems

Second order equations are the basis of analysis of mechanical and electrical systems. We'll build this important subject up slowly, starting with a simple mechanical system.

A spring is attached to a wall and a cart:



We set up the coordinate system so that at x = 0 the spring is relaxed, which means that it is exerting no force. This is called the **equilibrium position.**

In addition to the spring, suppose that there is another force acting on the cart – an **external force**, maybe wind blowing on a sail attached to it, maybe gravity, or some other force. Then

$$m\ddot{x} = F_{\text{spr}} + F_{\text{ext}}$$

The spring force is characterized by the fact that it depends only on position. In fact:

$$\begin{aligned} &\text{if } x > 0, \qquad F_{\text{spr}}(x) < 0 \\ &\text{if } x = 0, \qquad F_{\text{spr}}(x) = 0 \\ &\text{if } x < 0, \qquad F_{\text{spr}}(x) > 0. \end{aligned}$$

The simplest way to model this behavior (and one which is valid in general for small x, by the tangent line approximation) is

$$F_{\text{spr}}(x) = -kx, \text{ where } k > 0$$

This is called **Hooke's Law** and k is called the **spring constant**.

Replacing F_{spr} by $-kx$ we get

$$m\ddot{x} + kx = F_{\text{ext}}.$$

Any real mechanical system also has friction. Friction takes many forms. It is characterized by the fact that it depends on the motion of the mass. We will suppose that it depends only on the velocity of the mass and not on its position. Often the damping is controlled by a device called a **dashpot**. This is a cylinder filled with oil, that a piston moves through. Door dampers and car shock absorbers often actually work this way. We write $F_{\text{dash}}(\dot{x})$ for the force exerted by the dashpot. It opposes the velocity:

$$\begin{array}{ll} \text{if } \dot{x} > 0, & F_{\text{dash}}(\dot{x}) < 0 \\ \text{if } \dot{x} = 0, & F_{\text{dash}}(\dot{x}) = 0 \\ \text{if } \dot{x} < 0, & F_{\text{dash}}(\dot{x}) > 0 \end{array}$$

The simplest way to model this behavior (and one which is valid in general for small $\dot{x}$, by the tangent line approximation) is

$$F_{\text{dash}}(\dot{x}) = -b\dot{x}, \text{ where } b > 0.$$

This is therefore called **linear damping** and b is called the **damping constant**.

Putting this together we get the differential equation for the displacement x of the mass from equilibrium is

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}. \quad (1)$$

Equation (1) will be a rich source of examples in the remainder of the course. Diagramatically this looks like: