

PROFESSOR: Hi everyone. So today, I'd like to talk about frequency response. And specifically, we're going to take a look at a couple different differential equations. And we're asked to graph the amplitude response for each equation. And you'll note that these equations a, b and c have varying amounts of damping. So for part a we're asked to plot the amplitude response for $x'' + 4x = F_0 \cos(\omega t)$.

For part b it's the same equation, however we have an x' term added. For part c, again we've increase the damping. So now we have $6x'$. And then lastly for part d, we'd like to discuss the resonance for each system. So I'll let you work out this problem, and I'll be back in a moment.

Hi everyone. Welcome back. So for part a, we're asked to graph the amplitude response to the differential equation $x'' + 4x = F_0 \cos(\omega t)$. And from a previous recitation, we already wrote down the particular response to this differential equation. So I'm just going to write down the particular response, which has the form $F_0 / (4 - \omega^2) \cos(\omega t)$.

Now the amplitude response is defined as a ratio, and specifically it's the ratio of the output amplitude of a differential equation to the input amplitude of a differential equation. So in the case at hand, we have the output is a sinusoidal function whose amplitude is $F_0 / (4 - \omega^2)$. So it's the output divided by the input. These are both amplitudes. And in our case, we have $F_0 / (4 - \omega^2)$. This is the output amplitude.

And the input amplitude is just F_0 . So we see when we compute this ratio the F_0 's divide out. And at the end of the day, we're left with $1 / (4 - \omega^2)$.

So I'm going to draw the amplitude response now. So we have ω . And we see that when ω is equal to 2, there's an asymptote. When ω is equal to 0, we have $1/4$. And specifically, we have this tent-like function. So this is the amplitude response.

So notice how when we drive the system with frequency two, the amplitude response goes to infinity. As a result, we call this frequency the resonant frequency. So this concludes part a.

For part b, we have a differential equation with damping now. And to compute the particular solution, we follow the standard procedure of first complexifying the right-hand side and then

using the exponential response formula.

So I'm just going to write down the particular solution. If we follow these steps, we find that it's the real part of the right-hand side complexified, which is $F_0 e^{i\omega t}$ divided by the characteristic polynomial evaluated at $i\omega$. And in this case, the characteristic polynomial p of s is $s^2 + s + 4$. p of $i\omega$ is then $4 - \omega^2 + i\omega$. And when we put the pieces together, we end up with a particular solution, which looks like the real part of $\frac{1}{4 - \omega^2 + i\omega} F_0 e^{i\omega t}$.

So we're asked to compute the amplitude response graph. And if we take a look at this, we see that the denominator here is really just a complex number. So we can convert it into the form of $r e^{i\phi}$.

Now, the amplitude response is defined as the ratio of the output divided by the input. And so the output amplitude is going to be the magnitude of this complex number. So as a result, the amplitude response is just the magnitude of $\frac{1}{4 - \omega^2 + i\omega}$. This is also sometimes referred to as the complex gain.

Moreover, this term right here contains two pieces of information. Not only does it contain the amplitude response, but it also contains the phase information. When we take the absolute value, we're throwing out the phase information, and we're just remembering the amplitude response.

So what is this amplitude response look like for this case? Well, we have $\frac{1}{4 - \omega^2 + i\omega}$ and it's the magnitude of this complex number, which is $\sqrt{4 - \omega^2 + \omega^2}$. So I just take the real part, square it, add it to the imaginary part squared, and square root the whole quantity.

Now there's a question of how to graph this. And we see that first off, the square root's an increasing function. And we see that we're $\frac{1}{\text{increasing function}}$. So there's a trick, which is to just look first at sketching this piece which is under the radical sign. And if you look at trying to maximize this function-- so finding the critical points-- we'd see that in this case, we have one maximum, to $4 - \omega^2 + \omega^2$. And this is at when ω equals the square root of $\frac{7}{2}$. Sorry. This is a minimum.

So when I go to sketch this now, we have ω , we have the amplitude response. Now, I'm going to draw in $\frac{7}{2}$ from our previous diagram. Now, the square root of $\frac{7}{2}$ is just below 2, so

square root of $\frac{7}{2}$. So we end up with a maximum at $\frac{7}{2}$, and then a decay to infinity. And again, this is going to be $\frac{1}{4}$ when ω is 0. So this is the peak amplitude response.

So note that in this case, by adding damping, what we've done is we no longer have an asymptote at ω equals 2. But we now have a finite amplitude, which occurs at ω equals the square root of $\frac{7}{2}$. So I'm just going to clean up the board, and I'll be back with part c in a second.

For part a, we have an amplitude response diagram, which looks like a tent function. And at 2, ω equals 2, we have a resonance. So the amplitude response diverges.

Just like to point out, I made a small error before. I forgot to include absolute values on the denominator here. The amplitude response function, it's always a positive quantity. We always throw out any phase information and leave that for the phase in the description of the response of the linear system. So the amplitude response is always positive.

For part b, we added dampening to the system. And we see that there's actually a peak point which is at the square root of $\frac{7}{2}$. And the amplitude response is bounded at this point, but it achieves a maximum. And then again it decays to infinity.

So I'd like now to take a look at part c. And in part c, we have the differential equation $x'' + 6x' + 4x = F_0 \cos(\omega t)$. And again, the amplitude response is going to equal $\frac{1}{|p(i\omega)|}$. And in this case, $p(i\omega)$ is going to be $4 - \omega^2 + 6i\omega$. Well, we still have the $4 - \omega^2$ term. Instead of x' , we now have $6x'$, which gives us $6i\omega$.

And then again, we want to take the absolute value of this complex number. And when we take the absolute value, we just get the sum of the real parts squared plus the sum of the imaginary parts squared, which in this case is going to be $36\omega^2 + (4 - \omega^2)^2$, the whole quantity squared rooted, and then we have $\frac{1}{\sqrt{36\omega^2 + (4 - \omega^2)^2}}$.

So now if we'd like to plot this function, we can still do the same trick and try to maximize or find the critical points of the denominator under the radical. And if we did this, in this case we would find that the only critical point is when ω is equal to 0. Secondly, if we look at ω going to infinity, we see that the denominator goes to infinity. So this whole quantity must go to 0.

So if I were to go back here to the amplitude response for part c, again, when ω is equal

to 0 it's going to start off at $1/4$. I've just argued that it goes to 0 as ω goes to infinity. And since there are no critical points, we must smoothly paste the function between the two. And in fact, it's always decreasing. So the amplitude response, in this case, is just a decreasing function.

So this concludes part c. And now I'll take a look at part d. Discuss the resonance for each system. So in part a, we had no damping. And we saw that there was a resonance at $\omega = 2$. And the resonance manifested itself in the amplitude response graph with a divergent asymptote at $\omega = 2$. So as you drive the system close to $\omega = 2$, the amplitude of the system starts to diverge.

In case two we introduced damping into the system. So we still have a very large amplitude response at $\omega = \sqrt{7/2}$, however it's no longer infinite. And then lastly, when we increased damping even further so we had the $6x$ dot term, the presence of a peak disappeared. And in fact, the amplitude response just monotonically decayed from $1/2$ to infinity. So just constantly decreased to 0.

So I'd just like to conclude there, and I'll see you next time.