Part II Problems and Solutions

Problem 1: *[Euler's method]* **(a)** *Write y for the solution to $y' = 2x$ with $y(0) = 0$. What is $y(1)$? What is the Euler approximation for $y(1)$, using 2 equal steps? 3 equal steps? What about n steps, where n can now be any natural number? (It will be useful to know that $1 + 2 + \cdots + (n-1) = n(n-1)/2$.) As $n \to \infty$, these approximations should converge to $y(1)$. Do they?*

(b) *In the text and in class it was claimed that for small h, Euler's method for stepsize h has an error which is at most proportional to h. The n-step approximation for $y(1)$ has $h = 1/n$. What is the exact value of the difference between $y(1)$ and the n-step Euler approximation? Does this conform to the prediction?*

Solution: $y = x^2$, so $y(1) = 1$.

Euler's method with stepsize h for this equation: $x_k = kh$, $y_{k+1} = y_k + 2x_k h$.

With $n = 2$, $h = 1/2$:

k	x_k	y_k	$m_k = -y_k$	hm_k
0	0	0	0	0
1	1/2	0	1	1/2
2	1	1/2		

With $n = 3$, $h = 1/3$:

k	x_k	y_k	$m_k = -y_k$	hm_k
0	0	0	0	0
1	1/3	0	2/3	2/9
2	2/3	2/9	4/3	4/9
3	1	2/3		

With n arbitrary, $h = 1/n$:

k	x_k	y_k	$m_k = 2x_k$	$m_k h$
0	0	0	0	0
1	h	0	$2h$	$2h^2$
2	$2h$	$2h^2$	$4h$	$4h^2$
3	$3h$	$2h^2 + 4h^2$	$6h$	$6h^2$
4	$4h$	$2h^2 + 4h^2 + 6h^2$	$8h$	$8h^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

So $y_n = 2(1 + 2 + \cdots + (n-1))h^2 = n(n-1)h^2$. With $h = 1/n$ this gives our estimate for $y(1)$: $n(n-1)/n^2 = (n-1)/n$. The limit of this as $n \to \infty$ is 1, which is good, and the error is $1/n$, which is exactly h.