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18.034 Honors Differential Equations
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LECTURE 21. STEP FUNCTIONS

Step and impulse signals: motivation. In the analysis of

$$P(D)y := y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = f$$

we often view the differential equation as an input-output system, as shown below.

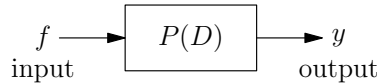


Figure 21.1. The input-output system.

Coefficients of $P(D)$ are parameters of the system, e.g. the spring constant in a spring-mass system or the resistance in the circuit analysis. In real life, we do not know the parameters in advance. Instead, we learn about the system by watching how it responds to various input signals.

The simpler the input signal is, the clearer we should expect the signature of the system parameters to be, and the easier we predict how the system will respond to other more complicated signals. The simplest is the null signal, which corresponds to the homogeneous equation. We study two other standard and very simple signals: the unit step function and the unit impulse. Here, we study step signals. Impulsive signals will be discussed in later lectures.

Step Signals. The *unit step function* or the *Heaviside** unit function is defined as

$$(21.1) \quad h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

It has a jump discontinuity at $t = 0$. The graph of $h(t)$ is given below.

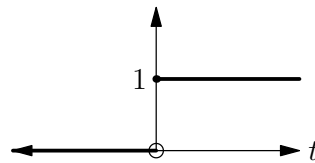


Figure 21.2. Graph of $h(t)$.

By definition,

$$h(t - c) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

Clearly,

$$\mathcal{L}[h(t - c)] = \int_0^\infty e^{-st} h(t - c) dt = \int_c^\infty e^{-st} dt = \frac{e^{-sc}}{s},$$

where $c \geq 0$.

The unit step function is used to describes various discontinuous inputs. If $f(t)$ is defined for $t \geq 0$ and $c \geq 0$, then $f(t)h(t - c)$ agrees with $f(t)$ for $t \geq c$ and is zero for $t < c$. If $0 \leq a < b$

*Oliver Heaviside (1850-1925), English engineer, mathematician, and physicist.

then $f(t)h(t - a) - f(t)h(t - b)$ agrees with $f(t)$ on the interval $[a, b]$ and is zero elsewhere. This function describes the effect of closing a switch at time $t = a$ and opening it at $t = b$. Lastly, if $c \geq 0$ the $f(t - c)h(t - c)$ is obtained by moving the graph of f over by c to the right. More precisely, if $g(t) = f(t - c)h(t - c)$ then $g(t)$ agrees with $f(t)$ for $t \geq c$ and is zero for $t < c$. These functions are shown in the figure below.

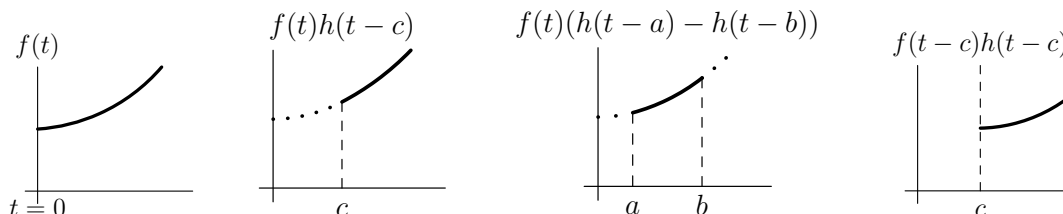


Figure 21.3. Discontinuous inputs.

One of the main uses of the unit step function is in connection with the t -shift property

$$\mathcal{L}[f(t - c)] = e^{-sc}F(s), \quad c \geq 0$$

which requires the condition $f(t) = 0$ for $t < 0$. Using the unit step function, alternatively, we may write this as

$$(21.2) \quad \mathcal{L}[f(t - c)h(t - c)] = e^{-sc}F(s), \quad c \geq 0.$$

Example 21.1. What function has Laplace transform $\frac{e^{-s}}{s^2}$?

SOLUTION. We know that $\mathcal{L}[t] = \frac{1}{s^2}$. By the t -shift property, then,

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = (t - 1)h(t - 1).$$

We use these results to solve the differential equation with a discontinuous input.

Example 21.2. Solve the initial value problem

$$y'' - 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f(t) = \begin{cases} 1, & t \in [0, 1), [2, 3), [4, 5) \\ 0, & t \in [1, 2), [3, 4), [5, \infty). \end{cases}$ The graph of $f(t)$ is given below.

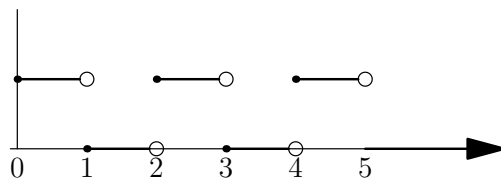


Figure 21.4. Graph of $f(t)$.

SOLUTION. Let $Y(s) = \mathcal{L}y$ and $F(s) = \mathcal{L}f$. Taking the transform,

$$(s^2 - 3s + 2)Y(s) = F(s), \quad Y(s) = \frac{F(s)}{(s - 1)(s - 2)}.$$

We write

$$f(t) = h(t) - h(t - 1) + h(t - 2) - h(t - 3) + h(t - 4) - h(t - 5),$$

and hence

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s}.$$

Alternatively,

$$F(s) = \int_0^1 e^{-st} dt + \int_2^3 e^{-st} dt + \int_4^5 e^{-st} dt.$$

Therefore,

$$Y(s) = \frac{1}{s(s-1)(s-2)}(1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s}).$$

By partial fractions, then,

$$\frac{1}{s(s-1)(s-2)} = \frac{1}{2} \frac{1}{s} - \frac{1}{s-1} + \frac{1}{2} \frac{1}{s-2} = \mathcal{L} \left[\frac{1}{2} - e^t + \frac{1}{2} e^{2t} \right],$$

and,

$$y(t) = \sum_{k=0}^5 y_0(t-k)h(t-k), \quad y_0(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t}.$$

Remark 21.3. It is easy to verify that $y_0(k) = y_0'(k) = 0$ for $k = 1, 2, \dots, 5$. That means, y and y' are continuous for $t \in [0, \infty)$. But, y'' is not continuous at $t = k$. In other words, y is a generalized solution.

Periodic functions. Let $f \in E$ be a periodic function of period $p > 0$. That is, $f(t+p) = f(t)$ for all t . The function

$$(21.3) \quad f_0(t) = f(t)h(t) - f(t)h(t-p)$$

agrees with $f(t)$ on the interval $t \in [0, p)$ and is zero elsewhere. In this sense, f_0 represents f on a single period, and f is the periodic extension of f_0 .

Since f is periodic, we can replace $f(t)$ in the second term of (21.3) by $f(t-p)$. Taking the transform, then gives $\mathcal{L}f_0 = \mathcal{L}f - e^{-ps}\mathcal{L}f$. Solving for $\mathcal{L}f$, we obtain

$$(21.4) \quad \mathcal{L}f = \frac{\mathcal{L}f_0}{1 - e^{-ps}}, \quad \text{where} \quad \mathcal{L}f_0 = \int_0^p e^{-st} f(t) dt.$$

Example 21.4. The square-wave function is defined by the equation

$$f(t) = \begin{cases} 1, & t \in [2n, 2n+1) \\ -1, & t \in [2n-1, 2n), \end{cases}$$

where $n \geq 0$ is an integer.

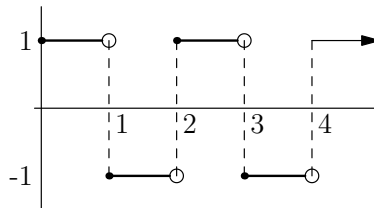


Figure 21. 5. Graph of the square-wave function.

Let

$$f_0(t) = h(t) - 2h(t-1) + h(t-2) = \begin{cases} 1, & t \in [0, 1) \\ -1, & t \in [1, 2). \end{cases}$$

Then, $\mathcal{L}[f_0(t)](s) = \frac{1 - 2e^{-s} + e^{-2s}}{s} = \frac{(1 - e^{-s})^2}{s}$. Therefore, by (21.4)

$$\mathcal{L}f = \frac{1}{s} \frac{(1 - e^{-s})^2}{1 - e^{-2s}} = \frac{1}{s} \frac{1 - e^{-s}}{1 + e^{-s}}.$$