18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #1

Due by Friday, February 13, 2009, by NOON.

1. (a) Verify that $y = x^a$ solves the differential equation $x^2 y'' = 2y$ if the constant a satisfies the equation $a^2 - a - 2 = 0$. Thus get the two solutions x^2 and x^{-1}. Note that the first is valid on the whole interval $-\infty < x < \infty$ but the second on $-\infty < x < 0$ or $0 < x < \infty$ only.
This behavior is typical for a broad class of linear homogeneous equations known as equations of *Euler type*. For this class the substitution $y = x^a$ always lead to an algebraic equation for a.

(b) The equation $x^2 y'''' = 2y''$ admits a solution $y = x^a$, where a is a nonzero constant. What are the possible values of a?

2. Suppose a function $y = f(x)$ satisfies the differential equation $dy = 4y \sin 2x \, dx$ and the initial condition $y(\pi) = e$. The purpose of this exercise is to find $y(\pi/6)$.
(a) Separate variables and integrate to obtain $\ln y = c - 2\cos 2x$, $y > 0$. By use of the initial condition show that $c = 3$, and then get $y(\pi/6)$.

(b) The initial condition $y(\pi) = e$ means that $x = \pi$ corresponds to $y = e$. Likewise, $x = \pi/6$ corresponds to $y = a$, where $a = y(\pi/6)$. Integrating between corresponding limits gives

$$\int_e^a \frac{dy}{y} = \int_\pi^{\pi/6} 4 \sin 2x \, dx.$$

Evaluate the definite integrals and solve the resulting equation for a.

(c) If $xdy + 3ydx = 0$ and $y(-\pi) = e$ you can't find $y(\pi)$. Why not?

3. Birkhoff-Rota, pp. 5, #3.

4. (a) Show that $y_1(x) = 0$ and $y_2(x) = x^{3/2}$ are solutions for $x \geq 0$ of the differential equation $y' = (3/2)y^{1/3}$.

(b) Discuss that all nonnegative solutions of the differential equation in part (a) starting at $(0, 0)$ lie between two solutions y_1 and y_2 and that the solution

$$y(x) = \begin{cases} 0 & \text{for} \quad x < c, \\ (x-c)^{3/2} & \text{for} \quad x \geq c \end{cases}$$

fills out the funnel between them.

5. Birkhoff-Rota, pp. 11, #7. (Typo. k is n.)

6. (The *Bernoulli* equation.) It is a differential equation of the form $y' + p(x)y = q(x)y^n$ with $n \neq 1$.
(a) Show that it becomes linear by the change of variable* $u = y^{1-n}$. (Hint. Begin by dividing both sides of the equation by y^n.)

(b) Solve the Bernoulli equation $y' + y = xy^3$ using the method in part (a).

*This trick was found by Leibniz in 1696.