18.034 FINAL EXAM
MAY 20, 2004

Name: ____

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Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 3 hours for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Date: Spring 2004.

Table of Laplace Transforms

	$y(t)$	$Y(s) = \mathcal{L}[y(t)]$
1.	$y^{(n)}(t)$	$s^n Y(s) - (y^{(n-1)}(0) + \cdots + s^{n-1} y(0))$
2.	t^n	$n!/s^{n+1}$
3.	$t^n y(t)$	$(-1)^n Y^{(n)}(s)$
4.	$\cos(\omega t)$	$s/(s^2+\omega^2)$
5.	$\sin(\omega t)$	$\omega/(s^2+\omega^2)$
6.	$e^{at} y(t)$	$Y(s-a)$
7.	$y(at),\ a > 0$	$\frac{1}{a} Y(s/a)$
8.	$S(t-t_0) y(t-t_0),\ t_0 \geq 0$	$e^{-st_0} Y(s)$
9.	$\delta(t-t_0),\ t_0 \geq 0$	e^{-st_0}
10.	$(S(t)y) * (S(t)z)$	$Y(s)Z(s)$
11.	$y(t),\ y(t+T) = y(t)$	$\frac{1}{1-e^{-sT}} \int_0^T e^{-st} y(t) dt$

Name: ____________________ **Problem 1**: __________ /10

Problem 1(10 points) Two objects of mass m are connected to a rigid base and to each other as shown on the previous page. The spring connecting each object to the base has constant k, and the spring connecting the objects to each other has constant $2k$. Denote by x_1 the displacement of the object on the left from equilibrium (displacement to the right = positive displacement). Denote by x_2 the displacement of the object on the right from equilibrium (displacement to the right = positive displacement). Denote $\omega = \sqrt{k/m}$.

(a)(5 points) Find a system of 2^{nd} order linear ODEs satisfied by x_1 and x_2 of the form,

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In other words, find the matrix A.

Solution: The force on Object 1 from the spring connected to the base is $-kx_1$, and the force from the spring connected to Object 2 is $2k(x_2 - x_1)$. The force on Object 2 from the spring connected to the base is $-kx_2$, and the force from the spring connected to Object 1 is $2k(x_1 - x_2)$. By Newton's Second Law,

$$\begin{cases} x_1'' &= \frac{-3k}{m}x_1 + \frac{2k}{m}x_2 \\ x_2'' &= \frac{2k}{m}x_1 + \frac{-3k}{m}x_2 \end{cases}$$

Therefore,

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3\omega^2 & 2\omega^2 \\ 2\omega^2 & -3\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Name: ______________________________ **Problem 1, contd.**

(b)(5 points) Introduce new variables $v_1 = x_1'$ and $v_2 = x_2'$. Find a system of 1st order linear ODEs satisfied by x_1, v_1, x_2 and v_2 of the form,

$$\begin{bmatrix} x_1' \\ v_1' \\ x_2' \\ v_2' \end{bmatrix} = B \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}.$$

In other words, find the matrix B.

Solution: By definition, $x_1' = v_1$ and $x_2' = v_2$. Therefore $v_1' = x_1''$ and $v_2' = x_2''$. From part (a),

$$\begin{cases} x_1' &= & & v_1 & & \\ v_1' &= & -3\omega^2 x_1 & + & 2\omega^2 x_2 & \\ x_2' &= & & & & v_2 \\ v_2' &= & 2\omega^2 x_1 & + & -3\omega^2 x_2 & \end{cases}$$

Therefore,

$$\begin{bmatrix} x_1' \\ v_1' \\ x_2' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3\omega^2 & 0 & 2\omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ 2\omega^2 & 0 & -3\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}.$$

Extra credit(2 points) What is the relationship of $p_A(\lambda)$ and $p_B(\lambda)$?

Solution: It can be computed directly that,

$$p_B(\lambda) = \lambda^4 + 6\lambda^2 + 5 = p_A(\lambda^2).$$

Another method is to observe that $p_{B^2}(\lambda^2) = p_B(\lambda)p_B(-\lambda)$. With respect to the ordered basis (x_2, x_2, v_1, v_2), the matrix for B^2 is the block matrix,

$$B^2 = \left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & A \end{array}\right].$$

Therefore, $p_{B^2}(\lambda) = (p_A(\lambda))^2$. Therefore,

$$p_B(\lambda)p_B(-\lambda) = (p_A(\lambda^2))^2.$$

For the pair of matrices A and B above, $p_B(\lambda) = p_B(-\lambda)$ so that $p_B(\lambda) = p_A(\lambda^2)$.

Name: ________________________ **Problem 2**: __________ /20

Problem 2(20 points) Consider the ODE,

$$y(t)' + \frac{2}{t}y(t) = 3e^{-t^3/3}, \quad t > 0$$

(a)(5 points) Find an integrating factor.

Solution: An integrating factor $u(t)$ is a solution of the separable ODE,

$$u' = \frac{2}{t}u, \quad \int \frac{du}{u} = 2\int \frac{dt}{t}.$$

Therefore $u = At^2$. Setting $A = 1$, $u(t) = t^2$ is an integrating factor.

(b)(10 points) Find the general solution.

Solution: Multiplying both sides of the equation by the integrating factor,

$$(t^2y(t))' = 3t^2e^{-t^3/3}dt.$$

Antidifferentiating both sides,

$$t^2y(t) = \int 3t^2e^{-t^3/3}dt.$$

Substituting $v = -t^3/3$, $dv = -t^2dt$, this is,

$$t^2y(t) = \int -3e^v dv = -3e^v + C = -3e^{-t^3/3} + C.$$

Therefore the general solution is,

$$y(t) = \frac{1}{t^2}(-3e^{-t^3/3} + C).$$

(c)(5 points) Find the unique solution that has an extension to a continuous function on $[0, \infty)$.

Solution: The issue is continuity at 0. Unless $C = 3$, $y(t)$ diverges as $t \to 0$. If $C = 3$, then

$$\lim_{t\to 0} \frac{3 - 3e^{-t^3/3}}{t^2} = \lim_{t\to 0} \frac{3 - 3(1 - t^3/3 + t^6/18 + \dots)}{t^2} = \lim_{t\to 0} \frac{t^3}{t^2} = 0.$$

So the unique solution is,

$$y(t) = \frac{1}{t^2}(-3e^{-t^3/3} + 3).$$

Name: ______________________________ **Problem 3**: ____________ /25

Problem 3(25 points) A basic solution pair of the homogeneous linear 2$^{\text{nd}}$ order ODE,

$$y'' + \frac{2t}{t^2-4}y' - 16\frac{1}{(t^2-4)^2}y = 0, \quad t > 2$$

is given by $\{y_1(t), y_2(t)\}$,

$$y_1(t) = \frac{t-2}{t+2}, \quad y_2(t) = \frac{t+2}{t-2}.$$

(a)(10 points) Compute the Wronskian $W[y_1, y_2](t)$.

Solution: The derivative of $y_1(t)$ is,

$$y_1'(t) = \frac{1}{(t+2)^2}(1(t+2) - 1(t-2)) = \frac{4}{(t+2)^2}.$$

Similarly, the derivative of $y_2(t)$ is,

$$y_2'(t) = \frac{1}{(t-2)^2}(1(t-2) - 1(t+2)) = \frac{-4}{(t-2)^2}.$$

Therefore the Wronskian is,

$$W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\frac{4}{(t-2)^2}\frac{t-2}{t+2} - \frac{4}{(t+2)^2}\frac{t+2}{t-2},$$

i.e.,

$$W[y_1, y_2](t) = \frac{-8}{t^2-4}.$$

Name: ______________________________ **Problem 3, contd.**

(b)(15 points) Use variation of parameters to find a particular solution of the inhomogeneous ODE,

$$y'' + \frac{2t}{t^2-4}y' - 16\frac{1}{(t^2-4)^2}y = 1.$$

Solution: By the method of variation of parameters, a particular solution is,

$$y_p(t) = \int_0^t K(s,t)f(s)ds,$$

where $K(s,t)$ is the Green's kernel,

$$K(s,t) = (y_1(s)y_2(t) - y_1(t)y_2(s))/W[y_1,y_2](s)$$

$$= -\tfrac{s^2-4}{8}\left(\tfrac{s-2}{s+2}\tfrac{t+2}{t-2} - \tfrac{t-2}{t+2}\tfrac{s+2}{s-2}\right)$$

$$= \tfrac{1}{8}\left((s+2)^2\tfrac{t-2}{t+2} - (s-2)^2\tfrac{t+2}{t-2}\right).$$

Of course $f(s) = 1$. Therefore a particular solution is,

$$y_p(t) = \tfrac{1}{8}\int_0^t \left((s+2)^2\tfrac{t-2}{t+2} - (s-2)^2\tfrac{t+2}{t-2}\right)ds$$

$$= \tfrac{1}{24}\tfrac{t-2}{t+2}\left((s+2)^3\right|_0^t - \tfrac{1}{24}\tfrac{t+2}{t-2}\left((s-2)^3\right|_0^t$$

$$= \tfrac{1}{24}(t-2)(t+2)((t+2)-(t-2)) - \tfrac{1}{3}y_1(t) + \tfrac{1}{3}y_2(t).$$

Therefore, a particular solution is,

$$y_p(t) = \frac{1}{6}(t^2-4).$$

Name: ______________________________ **Problem 4**: ____________ /15

Problem 4(15 points) Using the method of undetermined coefficients and the exponential shift rule, find a particular solution of the inhomogeneous linear 2$^{\text{nd}}$ order ODE,

$$y'' + 5y' + 6y = -4te^{-3t}.$$

Solution: In operator form, this is,

$$(D^2 + 5D + 6)y(t) = -4te^{-3t}.$$

The guess is that a particular solution is of the form $y(t) = e^{-3t}g(t)$. By the exponential shift rule,

$$(D^2 + 5D + 6)e^{-3t}g(t) = e^{-3t}((D-3)^2 + 5(D-3) + 6)g(t) = (D^2 - D)g(t).$$

Therefore,

$$(D^2 - D)g(t) = -4t.$$

By the method of undetermined coefficients, $g(t)$ is a general degree 2 polynomial,

$$g(t) = a_2t^2 + a_1t + a_0.$$

Plugging in,

$$(D^2 - D)g(t) = (-2a_2)t + (2a_2 - a_1).$$

Therefore a solution is, $a_2 = 2$, $a_1 = 4$, and $a_0 = 0$ (or any constant actually). So a particular solution is,

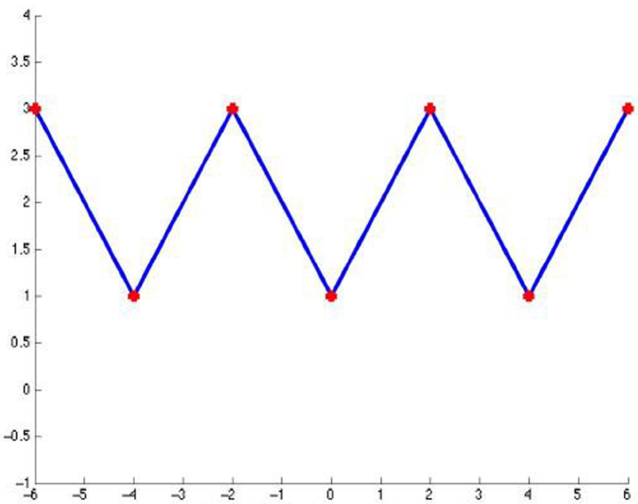
$$y_p(t) = (2t^2 + 4t)e^{-3t}.$$

Name: ______________________ **Problem 5**: __________ /20

Problem 5(20 points) On the interval $[0, 2)$, let $f(t) = t + 1$. Denote by $\widetilde{f}(t)$ the even extension of $f(t)$ as a periodic function of period 4. Denote by FCS$[\widetilde{f}]$ the Fourier cosine series of $\widetilde{f}(t)$.

(a)(5 points) Graph FCS$[\widetilde{f}]$ on the interval $[-4, 4]$. Make special note of all discontinuities and the *actual value* of FCS$[\widetilde{f}]$ at these points.

Solution: A graph of the solution is given below. There are no discontinuities.



Fourier Cosine Series from Problem 5

Name: ________________________________ **Problem 5, contd.**

(b)(10 points) An orthonormal basis for the even periodic functions of period 4 is,

$$\phi_0(t) = \frac{1}{2}, \quad \phi_n(t) = \frac{1}{\sqrt{2}}\cos(n\pi t/2), \ n = 1, 2, 3, \dots$$

Compute the Fourier coefficients,

$$a_n = \langle \widetilde{f}, \phi_n \rangle = \int_{-2}^{2} \widetilde{f}(t)\phi_n(t)dt,$$

and express your answer as a Fourier cosine series,

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{2}}\cos(n\pi t/2).$$

Don't forget to compute a_0.

Solution: First of all,

$$a_0 = 2\int_0^2 \frac{1}{2}f(t)dt = \int_0^2 t + 1dt = \left(\frac{t^2}{2} + t\right|_0^2 = 4.$$

For each integer $n > 0$,

$$a_n = 2\int_0^2 \frac{1}{\sqrt{2}}f(t)\cos(n\pi t/2)dt = \sqrt{2}\int_0^2 (t+1)\cos(n\pi t/2)dt.$$

Of course,

$$\int_0^2 \cos(n\pi t/2)dt = \left(\frac{2}{n\pi}\sin(n\pi t/2)\right|_0^2 = 0.$$

Compute,

$$\int_0^2 t\cos(n\pi t/2)dt,$$

using integration by parts,

$$\begin{array}{rclcrcl} u &=& t, & dv &=& \cos(n\pi t/2)dt \\ du &=& dt, & v &=& \frac{2}{n\pi}\sin(n\pi t/2) \end{array}$$

So,

$$\int_0^2 t\cos(n\pi t/2) = \int udv = uv - \int vdu =$$

$$\left(\tfrac{2}{n\pi}t\sin(n\pi t/2)\right|_0^2 - \int_0^2 \tfrac{2}{n\pi}\sin(n\pi t/2)dt =$$

$$0 + \left(\tfrac{4}{n^2\pi^2}\cos(n\pi t/2)\right|_0^2 =$$

$$\tfrac{4}{n^2\pi^2}((-1)^n - 1).$$

Therefore,

$$a_n = \frac{4\sqrt{2}}{n^2\pi^2}((-1)^n - 1).$$

In particular, $a_n = 0$ if n is even. And if n is odd, then,

$$a_n = \frac{-8\sqrt{2}}{n^2\pi^2}.$$

Therefore the Fourier cosine series is,

$$\mathrm{FCS}[\widetilde{f}] = 2 - \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos((2m+1)\pi t/2).$$

Extra credit(3 points) Plug in $t = 0$ to get a formula for the series,

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Solution: Because $\widetilde{f}(t)$ is continuous and piecewise continuously differentiable, the Fourier cosine series converges pointwise to $\widetilde{f}(t)$. In particular, $\widetilde{f}(t) = 1$. Therefore,

$$1 = \mathrm{FCS}[\widetilde{f}](0) = 2 - \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}1.$$

Solving for the series gives,

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Name: ______________________ **Problem 6**: __________ /25

Problem 6(25 points) Let $f(t)$ be the piecewise continuous function,

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ e^{-3(t-1)}, & t \geq 1 \end{cases}$$

Let $y(t)$ be the continuously differentiable and piecewise twice-differentiable solution of the following IVP,

$$\begin{cases} y'' + 5y' + 6y = f(t), \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$\mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t) dt.$$

(a)(5 points) Compute the Laplace transform of the IVP and use this to find an equation that $Y(s)$ satisfies.

First of all, $f(t) = S(t-1)g(t-1)$ where,

$$g(t) = e^{-3t}.$$

Therefore,

$$\mathcal{L}[f(t)] = \mathcal{L}[S(t-1)g(t-1)] = e^{-s}\mathcal{L}[g(t)] = \frac{e^{-s}}{s+3}.$$

Also, $\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s)$ and $\mathcal{L}[y''(t)] = s^2Y(s) - (sy(0) + y'(0)) = s^2Y(s)$. Therefore,

$$(s^2 + 5s + 6)Y(s) = \frac{e^{-s}}{s+3}.$$

(b)(10 points) Solve the equation fo $Y(s)$ and find the partial fraction decomposition of your answer. Use the Heaviside cover-up method to simplify the partial fraction decomposition.

Solution: The quadratic polynomial $s^2 + 5s + 6$ factors as $(s+3)(s+2)$. Therefore,

$$Y(s) = e^{-s}\frac{1}{(s+2)(s+3)^2}.$$

The partial fraction decomposition is,

$$\frac{1}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{(s+3)^2} + \frac{C}{s+3}.$$

By the Heaviside cover-up method, $A = 1$ and $B = -1$. Finally, plugging in $s = -1$ gives,

$$\frac{1}{1 \times 2^2} = \frac{1}{1} + \frac{-1}{2^2} + \frac{C}{2},$$

i.e., $C = -1$. Therefore,

$$Y(s) = e^{-s}\frac{1}{s+2} - e^{-s}\frac{1}{(s+3)^2} - e^{-s}\frac{1}{s+3}.$$

Name: ______________________________ **Problem 6, contd.**

(c)(10 points) Find $y(t)$ by computing the inverse Laplace transform.

Solution: First of all, if $\mathcal{L}^{-1}[Z(s)] = z(t)$, then $\mathcal{L}^{-1}[e^{-s}Z(s)] = S(t-1)z(t-1)$. Therefore,

$$\mathcal{L}^{-1}[Y(s)] = S(t-1)e^{-2(t-1)} - S(t-1)(t-1)e^{-3(t-1)} - S(t-1)e^{-3(t-1)}.$$

Simplifying, this is,

$$y(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ e^{-2(t-1)} - te^{-3(t-1)}, & t > 1 \end{cases}$$

Question:(Not to be answered) Is there a simpler solution than using the Laplace transform? If so, you can use this to double-check your answer.

Solution: This question was not to be answered. The point is that, after substituting $\tau = t - 1$ and $z(\tau) = y(\tau + 1)$, the original IVP is equivalent to the IVP,

$$\begin{cases} z''(\tau) + 5z'(\tau) + 6z(\tau) = e^{-3\tau}, \\ z(0) = 0, \\ z'(0) = 0 \end{cases}$$

Denoting $Z(s) = \mathcal{L}[z(\tau)]$, the equation for $Z(s)$ is,

$$(s^2 + 5s + 6)Z(s) = \frac{1}{s+3}.$$

The solution is, as before,

$$Z(s) = \frac{1}{s+2} - \frac{1}{(s+3)^2} - \frac{1}{(s+3)}.$$

Taking the inverse Laplace transform,

$$z(\tau) = e^{-2\tau} - (\tau + 1)e^{-3\tau}.$$

Therefore,

$$y(t) = z(t-1) = \begin{cases} 0, & 0 \leq t \leq 1, \\ e^{-2(t-1)} - te^{-3(t-1)}, & t > 1 \end{cases}$$

Name: ______________________________ **Problem 7**: ____________ /10

Problem 7 Let A be the real 3×3 matrix,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a)(3 points) Compute the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$.

Solution: Because this matrix is upper triangular,

$$p_A(\lambda) = (\lambda - 2)^2(\lambda + 1).$$

(b)(7 points) For each eigenvalue, find an eigenvector (*not a generalizaed eigenvector*).

Solution: For the eigenvalue $\lambda_1 = -1$, an eigenvector is a nullvector of,

$$A + I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

For a nullvector, from the second and third row, the last coordinate must be zero. Then from the first row, a nullvector is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 2$, an eigenvector is a nullvector of,

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A nullvector is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Name: ________________________________ **Problem 7, contd.**

Extra credit(3 points) For one of the eigenvalues, the eigenspace is deficient. Find a generalized eigenvector that is not an eigenvector.

Solution: For $\lambda_2 = 2$, the eigenspace is deficient. A generalized eigenvector is a nullvector of $(A + 2I)^2$ that is not a nullvector of $A + 2I$. Now,

$$(A + 2I)^2 = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore a generalized eigenvector is,

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Notice that,

$$(A - 2I)\mathbf{v}_3 = \mathbf{v}_2.$$

Therefore, with respect to the basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, the matrix is in Jordan normal form,

$$[A]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Name: ______________________________ **Problem 8**: ____________ /35

Problem 8(35 points) The linear system $\mathbf{x}' = A\mathbf{x}$ is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a)(5 points) Compute Trace(A), $\det(A)$, the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$ and the eigenvalues of A.

Solution: First of all, Trace$(A) = -5$ and $\det(A) = 0(-5) - (-1)6 = 6$. Therefore,

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 + 5\lambda + 6.$$

This factors as $(\lambda + 2)(\lambda + 3)$. So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -2$.

(b)(10 points) For each eigenvalue (or for one eigenvalue from each complex conjugate pair), find an eigenvector.

Solution: For the eigenvalue $\lambda_1 = -3$, an eigenvector is a nullvector of,

$$A + 3I = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}.$$

So an eigenvector is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = -2$, an eigenvector is a nullvector of,

$$A + 2I = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}.$$

So an eigenvector is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Name: ______________________________ **Problem 8, contd.**

(c)(5 points) Find the general solution of the linear system of ODEs. Write your answer in the form of a solution matrix $X(t)$ whose column vectors are a basis for the solution space.

Solution: A basis for the solution space is given by,

$$\mathbf{v}_1 e^{\lambda_1 t} = \begin{bmatrix} e^{-3t} \\ 3e^{-3t} \end{bmatrix},$$

together with,

$$\mathbf{v}_2 e^{\lambda_2 t} = \begin{bmatrix} e^{-2t} \\ 2e^{-2t} \end{bmatrix}.$$

Therefore the solution matrix is,

$$X(t) = \begin{bmatrix} e^{-3t} & e^{-2t} \\ 3e^{-3t} & 2e^{-2t} \end{bmatrix}.$$

(d)(5 points) Compute the exponential matrix,

$$\exp(tA) = X(t)X(0)^{-1}.$$

Solution: For $t = 0$, the solution matrix is,

$$X(0) = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

Therefore,

$$X(0)^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}.$$

So the exponential matrix is,

$$\exp(tA) = X(t)X(0)^{-1} = \begin{bmatrix} e^{-3t} & e^{-2t} \\ 3e^{-3t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix},$$

which is,

$$\exp(tA) = \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & e^{-3t} - e^{-2t} \\ -6e^{-3t} + 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}.$$

Name: ______________________________ **Problem 8, contd.**

(e)(10 points) Denote by $\mathbf{f}(t)$ the vector-valued function,

$$\mathbf{f}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}.$$

Denote by $\mathbf{x}_0$ the column vector,

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the following IVP write down the solution in terms of the matrix exponential.

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases}$$

Compute the entries of the constant term vector and the integrand column vector, but do not evaluate the integrals.

Solution: The solution to the IVP is,

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 + \exp(tA)\int_0^t \exp(-sA)\mathbf{f}(s)ds.$$

First of all,

$$\exp(tA)\mathbf{x}_0 = \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & e^{-3t} - e^{-2t} \\ -6e^{-3t} + 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-3t} - e^{-2t} \\ 3e^{-3t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^{-3t} - e^{-2t} \\ 3e^{-3t} - 2e^{-2t} \end{bmatrix}.$$

Next,

$$\exp(-sA) = \begin{bmatrix} -2e^{3s} + 3e^{2s} & e^{3s} - e^{2s} \\ -6e^{3s} + 6e^{2s} & 3e^{3s} - 2e^{2s} \end{bmatrix},$$

therefore,

$$\exp(-sA)\mathbf{f}(s) = \begin{bmatrix} -2e^{3s} + 3e^{2s} & e^{3s} - e^{2s} \\ -6e^{3s} + 6e^{2s} & 3e^{3s} - 2e^{2s} \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} = \begin{bmatrix} -2se^{3s} + 3se^{2s} \\ -6se^{3s} + 6se^{2s} \end{bmatrix}.$$

So, altogether,

$$\mathbf{x}(t) = \begin{bmatrix} e^{-3t} - e^{-2t} \\ 3e^{-3t} - 2e^{-2t} \end{bmatrix} + \exp(tA)\int_0^t \begin{bmatrix} -2se^{3s} + 3se^{2s} \\ -6se^{3s} + 6se^{2s} \end{bmatrix} ds.$$

Name: ______________________________ **Problem 9**: ____________ /40

Problem 9(40 points) Consider the following nonlinear, autonomous planar system,

$$\begin{cases} x' &= 12x(y-1) \\ y' &= 2y(x+y-2) \end{cases}$$

(a)(5 points) Find all equilibrium points.

Solution: The equilibrium points are,

$$p_1 = (0,0), \quad p_2 = (0,2), \quad p_3 = (1,1).$$

(b)(5 points) Determine the linearization at each equilibrium point.

Solution: The Jacobian matrix is,

$$\left[\frac{\partial F_i}{\partial x_j}\right] = \begin{bmatrix} 12y-12 & 12x \\ 2y & 2x+4y-4 \end{bmatrix}.$$

So for $p_1 = (0,0)$, the linearization is,

$$A_1 = \begin{bmatrix} -12 & 0 \\ 0 & -4 \end{bmatrix},$$

for $p_2 = (0,2)$, the linearization is,

$$A_2 = \begin{bmatrix} 12 & 0 \\ 4 & 4 \end{bmatrix},$$

and for $p_3 = (1,1)$, the linearization is,

$$A_3 = \begin{bmatrix} 0 & 12 \\ 2 & 2 \end{bmatrix}.$$

Name: ________________________________ **Problem 9, contd.**

(c)(15 points) For each linearization, determine the eigenvalues. If the eigenvalues are complex conjugates, determine the rotation (clockwise in/out, counterclockwise in/out). If the eigenvalues are real, determine roughly the eigenvectors and the type of the local phase portrait.

Solution: Because A_1 is diagonal, clearly $p_{A_1}(\lambda) = (\lambda+12)(\lambda+4)$. For the eigenvalue $\lambda_{1,1} = -12$, the eigenvector is,

$$\mathbf{v}_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and for the eigenvalue $\lambda_{1,2} = -4$, the eigenvector is,

$$\mathbf{v}_{1,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In fact, for $\lambda_{1,1} = -12$, the orbits whose tangent directions at p_1 are $\pm\mathbf{v}_{1,1}$ are,

$$\begin{cases} y = 0, & x > 0, \\ y = 0, & x < 0 \end{cases}$$

And for $\lambda_{1,2} = -4$, the orbits whose tangent directions at p_1 are $\pm\mathbf{v}1,2$ are,

$$\begin{cases} x = 0, & 0 < y < 2, \\ x = 0, & y < 0 \end{cases}$$

In particular, p_1 is an attractor and locally the orbital portrait is a stable node.

Because A_2 is lower triangular, clearly $p_{A_2}(\lambda) = (\lambda - 12)(\lambda - 4)$. For the eigenvalue $\lambda_{2,1} = 12$, the eigenvector is,

$$\mathbf{v}_{2,1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and for the eigenvalue $\lambda_{2,2} = -4$, the eigenvector is,

$$\mathbf{v}_{2,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In fact for $\lambda_{2,2} = -4$, the orbits whose tangent directions at p_2 are $\pm\mathbf{v}_{2,2}$ are,

$$\begin{cases} x = 0, & y > 2, \\ x = 0, & 0 < y < 2 \end{cases}$$

In particular, p_2 is a repellor and locally the orbital portrait is an unstable node.

For A_3,

$$p_{A_3}(\lambda) = \lambda^2 - \mathrm{Trace}(A_3)\lambda + \det(A_3) = \lambda^2 - 2\lambda - 24 = (\lambda - 6)(\lambda + 4).$$

For the eigenvalue $\lambda_{3,1} = -4$, the eigenvector is,

$$\mathbf{v}_{3,1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

and for the eigenvalue $\lambda_{3,2} = 6$, the eigenvector is,

$$\mathbf{v}_{3,2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

In particular, p_3 is unstable and locally the orbital portrait is a saddle.

Name: ________________________________ **Problem 9, contd.**

For the following 2 parts, **please sketch your answer on the graph on the following page**.

(d)(5 points) Using a dashed line, sketch the x-nullcline and y-nullcline. Draw a few representative arrows indicating the direction of the orbits on the nullcline on each side of each equilibrium point.

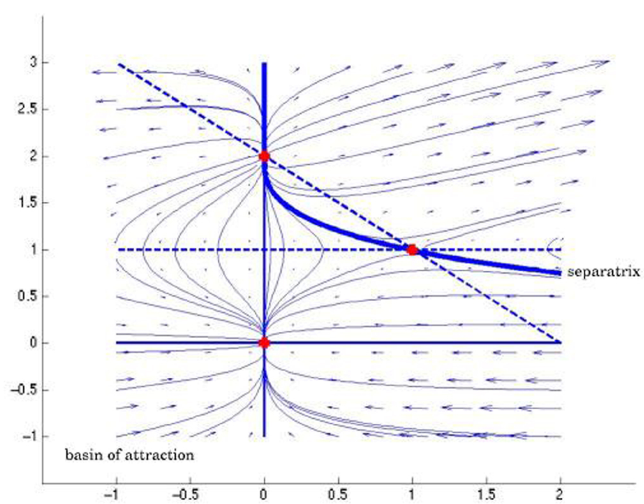
Solution: A sketch of the orbital portrait with the nullclines is on the course webpage. The x-nullcline is the union of the y-axis and the line $y = 1$. On the y-axis, the y-component of the direction field is positive for $y < 0$, is negative for $0 < y < 2$, and is positive for $y > 2$. On the line $y = 1$, the y-component of the direction field is negative for $x < 1$ and is positive for $x > 1$.

The y-nullcline is the union of the x-axis and the "antidiagonal" line $x + y = 2$. On the x-axis, the x-component of the direction field is positive for $x < 0$ and is negative for $x > 0$. On the line $x + y = 2$, the x-component of the direction field is negative for $x < 0$, is positive for $0 < x < 1$, and is negative for $x > 1$.

(e)(10 points) Sketch the phase portrait. Label all equilibrium points. For each equilibrium point, sketch a few orbits. In particular, for each saddle sketch each orbit whose limit or inverse limit is the equilibrium point.

There is one basin of attraction. Use bold lines to indicate each (rough) separatrix bounding this basin of attraction. Your sketch should just be a rough sketch, but it should be qualitatively correct.

Solution: A sketch of the orbital portrait is given below. The basin of attraction of p_1 is bounded by the curve made up of 3 orbits. The first piece of the curve is the orbit $x = 0, y > 2$. The second piece of the curve is the orbit whose inverse limit is $p_2 = (0, 2)$ and whose forward limit is $p_3 = (1, 1)$. This is the orbit whose tangent direction at p_3 is $\mathbf{v}_{3,1}/||\mathbf{v}_{3,1}||$. The third piece of the curve is the orbit whose forward limit is p_3 and whose tangent direction at p_3 is $-\mathbf{v}_{3,1}/||\mathbf{v}_{3,1}||$. The inverse limit is asymptotic to the orbit $y = 0, x > 0$. All points to the left and/or below this curve are in the basin of attraction of p_1.



Orbital Portrait from Final Exam

Name: ______________________ **Problem 10**: __________ /10

Problem 10, Extra Credit Problem(10 points) Let $a(t)$ and $b(t)$ be continous functions on an interval (c,d). Let $\{y_1, y_2\}$ be a basic solution pair on this interval of the ODE,

$$y'' + a(t)y' + b(t)y = 0.$$

Prove that between every two zeroes of y_1 there is a zero of y_2 (and vice versa).

Solution: Consider the zeroes of y_1 on (c,d). Let t be a zero of y_1. Note that y_1 is not an accumulation point of the zeroes of y_1: if t is an accumulation point, then $y_1(t) = y_1'(t) = 0$, which implies that $W[y_1, y_2](t) = 0$ contrary to Abel's theorem/the definition of a basic solution pair. In particular, if there is a zero $t' < t$ of y_1, then there is a maximal zero less than t. And if there is a zero $t' > t$ of y_1, then there is a minimal zero greater than t. Thus to prove that between every two zeroes of y_1 there is a zero of y_2, it suffices to prove this for every pair of zeroes $t < t'$ such that t' is the minimal zero greater than t.

By hypothesis, y_1 is nonzero on (t, t'). Without loss of generality, assume that y_1 is positive on (t, t'); up to replacing y_1 by $-y_1$, this is true. Then $y_1'(t) > 0$ and $y_1'(t') < 0$. By Abel's theorem, $W[y_1, y_2](t)$ is nonzero on $[t, t']$. Up to replacing y_2 by $-y_2$, assume that $W[y_1, y_2](t)$ is positive on $[t, t']$. Then,

$$W[y_1, y_2](t) = -y_1'(t)y_2(t) > 0,$$

which implies that $y_2(t) < 0$. And,

$$W[y_1, y_2](t') = -y_1'(t')y_2(t') > 0,$$

which implies that $y_2(t') > 0$. Therefore, by the intermediate value theorem, there exists t'' with $t < t'' < t'$ such that $y_2(t'') = 0$. So between any two zeroes of y_1, there is a zero of y_2. By the same argument, between any two zeroes of y_2, there is a zero of y_1.