18.034 EXAM 2
MARCH 17, 2004

Name: ______________________________________

Problem 1:	__________	/30
Problem 2:	__________	/20
Problem 3:	__________	/25
Problem 4:	__________	/25
Total:	__________	/100

Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Date: Spring 2004.

Name: ______________________________ **Problem 1**: ____________ /30

Problem 1(30 points) A driven, damped harmonic oscillator satisfies the following ODE,

$$y'' + 2by' + \omega^2 y = F\cos(\omega t),$$

where b, ω and F are positive real numbers.

(a)(10 points) Using the method of undetermined coefficients, find a solution of the form $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$.

Solution A particular solution is the real part of the complex-valued solution of,

$$\widetilde{y}'' + 2b\widetilde{y}' + \omega^2\widetilde{y} = Fe^{i\omega t}.$$

The complex number $i\omega$ is not a root of the characteristic polynomial. Therefore, we guess that $\widetilde{y} = e^{i\omega t}A$, for some complex number A. Substituting in gives,

$$e^{i\omega t}(-\omega^2 + 2ib\omega + \omega^2)A = e^{i\omega t}F.$$

Therefore $A = -iF/2b\omega$. This gives,

$$\widetilde{y}(t) = \frac{F}{2b\omega}\sin(\omega t) - i\frac{F}{2b\omega}\cos(\omega t).$$

So a particular solution is the real part,

$$y_d(t) = \frac{F}{2b\omega}\sin(\omega t).$$

(b)(10 points) Let R be a positive real number, $R \leq F/(2b\omega)$ (this guarantees that $y_d(t) = R$ for some $t > 0$). There is a multi-valued function $T = T(\omega)$ for the set of positive numbers where $y(T) = R$. This can be made into a single-valued function T_n by specifying that T_n is the n^{th} smallest positive number such that $y(T) = R$. So T_1 is the smallest positive number such that $y(T_1) = R$, T_2 is the smallest positive number greater than T_1 such that $y(T_2) = R$, etc.

For at least one choice of $n > 0$, find a formula for $T_n(\omega)$.

Solution The solution below is only valid if ω is *strictly* less than $F/(2bR)$. The functions T_n, $n \geq 2$, are discontinuous at $\omega = F/(2bR)$. Therefore, assume that $0 < \omega < F/(2bR)$.

The positive number T equals T_n for some n if

$$\frac{F}{2b\omega}\sin(\omega T) = R.$$

Solving for T gives,

$$T(\omega) = \frac{1}{\omega}\sin^{-1}(2b\omega R/F).$$

The different values T_1, T_2, etc. correspond to the different positive branches of $\sin^{-1}(2b\omega R/F)$. In particular, defining $\sin^{-1}(\theta)$ to be the usual branch,

$$T_1(\omega) = \frac{1}{\omega}\sin^{-1}(2b\omega R/F), \quad 0 < T_1(\omega) < \frac{\pi}{2\omega},$$

the formula for T_n is,

$$T_n(\omega) = \begin{cases} \frac{1}{\omega}((n-1)\pi + \sin^{-1}(2b\omega R/F)), & \text{if } n \text{ is odd} \\ \\ \frac{1}{\omega}((n-1)\pi - \sin^{-1}(2b\omega R/F)), & \text{if } n \text{ is even} \end{cases}$$

(c)(10 points) Suppose that b, F and R are fixed, but ω is allowed to vary,

$$0 < \omega \leq \frac{F}{2bR}.$$

Write $T = T_n$ for some positive integer n. Let ω be a critical point of $T(\omega)$. Prove there is an equation,

$$\omega T(\omega) = \alpha,$$

where α is a real number independent of b, F and R. Moreover, find an equation that α satisfies. (**Remark** You will see there are many choices for α. You are not responsible for "matching" choices of α to choices of n; just write down an equation that α satisfies).

Solution Implicitly differentiating the following relation with respect to ω,

$$\frac{1}{\omega}\sin(\omega T) = \frac{2bR}{F},$$

gives the following,

$$\frac{1}{\omega^2}(\omega\cos(\omega T)(T + \omega T') - \sin(\omega T)) = 0.$$

Equivalently, this is,

$$T + \omega T' = \frac{1}{\omega}\tan(\omega T).$$

At a critical point, $T'(\omega) = 0$, giving the equation,

$$\omega T = \tan(\omega T).$$

So $\omega T(\omega) = \alpha$, where α is a positive solution of the equation,

$$\alpha = \tan(\alpha).$$

Observe that the corresponding values of ω and T are,

$$\omega = \frac{F}{2bR}\sin(\alpha), \quad T = \frac{2bR}{F}\alpha\csc(\alpha).$$

Therefore, to be perfectly accurate, $\omega T = \alpha$ only gives a critical point of T_n if $n = 2m+1$ is odd, in which case the corresponding value of α is the unique solution,

$$\alpha = \tan(\alpha), \quad 2m\pi < \alpha \leq \left(2m + \frac{1}{2}\right)\pi.$$

(But this is more detail than you were asked to give).

Remark I was asked why the functions T_n do not have a critical point if n is even. The answer is related to the discontinuity of T_n at $F/(2bR)$. For n even, the function T_n does have a continuous extension to the interval $(0, \frac{F}{2bR}]$ (although it does not agree with the definition given above). The only extremal point of T_n is the global minimum of T_n, which occurs at $\omega = F/(2bR)$.

Extra credit(5 points) Prove the solutions of your equation for α give local minima of $T(\omega)$.

Solution The implicit differentiation above gives,

$$T + \omega T' = \frac{1}{\omega}\tan(\omega T).$$

Implicitly differentiating once again gives,

$$2T' + \omega T'' = \frac{1}{\omega^2}(\omega\sec^2(\omega T)(T + \omega T') - \tan(\omega T)) = \frac{1}{\omega^2\cos^2(\omega T)}(\omega(T + \omega T') - \sin(\omega T)\cos(\omega T)).$$

Plugging in $\omega T = \alpha$ and $T' = 0$ gives,

$$T'' = \frac{1}{\omega^3\cos^2(\alpha)}(\alpha - \sin(\alpha)\cos(\alpha)).$$

Because $\alpha = \tan(\alpha)$, $\sin(\alpha)\cos(\alpha) = \alpha\cos^2(\alpha)$. So the equation simplifies to,

$$T'' = \frac{1}{\omega^3}\alpha\tan^2(\alpha).$$

In particular, this is positive. So ω gives a local minimum of $T(\omega)$.

Name: ____ **Problem 2**: ________________ /20

Problem 2(20 points) In each case below, $y_1(t)$, $y_2(t)$ is a pair of solutions of a real, constant coefficient, linear homogeneous ODE in normal form. Determine the least degree of this ODE, and write down the ODE of this degree that the pair satisfies. (**Hint**: In each case, write down each nonzero solution as the real or imaginary part of $e^{\lambda t}g(t)$ where $g(t)$ is a polynomial. What does the degree of $g(t)$, and the vanishing/nonvanishing of the imaginary part of λ tell you about the characteristic equation of the ODE? Remember, the ODE is a *real* ODE.)

(a)(5 points) $y_1(t) = 0$, $y_2(t) = e^t$.

Solution The first equation, 0, is a solution of ANY homogeneous linear ODE. So this imposes no condition. The second condition imposes that 1 is a root of the characteristic polynomial. Therefore the minimal degree is 1, and the corresponding ODE is,

$$y' - y = 0.$$

(b)(5 points) $y_1(t) = e^{-t}$, $y_2(t) = e^{-2t}$.

Solution From the two equations, both -1 and -2 are roots of the characteristic polynomial. Therefore the characteristic polynomial is divisible by $(z+1)(z+2) = z^2 + 3z + 2$. So the minimal degree is 2, and the corresponding ODE is,

$$y'' + 3y' + 2y = 0.$$

(c)(5 points) $y_1(t) = t$, $y_2(t) = e^t$.

Solution From the first solution, 0 is a double root of the characteristic polynomial. From the second solution, 1 is a root of the characteristic polynomial. Therefore the characteristic polynomial is divisible by $z^2(z-1) = z^3 - z^2$. So the minimal degree is 3, and the corresponding ODE is,

$$y''' - y'' = 0.$$

(d)(5 points) $y_1(t) = \sin(2t)$, $y_2(t) = \cos(3t)$.

Solution From the first solution, $2i$ is a root of the characteristic polynomial. Therefore the complex conjugate, $-2i$ is also a root. From the second solution, $3i$ and $-3i$ are roots. Therefore the characteristic polynomial is divisible by $(z^2+4)(z^2+9)$. So the minimal degree is 4, and the corresponding ODE is,

$$y'''' + 13y'' + 36y = 0.$$

Name: ____ **Problem 3**: _________________ /25

Problem 3(25 points) For a certain linear ODE in normal form Ly a basic solution set of $Ly = 0$ is given by,

$$y_1(t) = e^{2t}, \quad y_2(t) = 2t^2 + 2t + 1.$$

(a)(10 points) Compute the Wronskian of this basic solution pair. Is your answer consistent with Abel's theorem?

Solution The derivatives of the functions are,

$$\begin{aligned} y_1(t) &= e^{2t}, & y_2(t) &= 2t^2 + 2t + 1, \\ y_1'(t) &= 2e^{2t}, & y_2'(t) &= 4t + 2 \end{aligned}$$

Therefore the Wronskian is

$$W[y_1, y_2](t) = (4t+2)e^{2t} - 2(2t^2 + 2t + 1)e^{2t} = -4t^2 e^{2t}.$$

For $t > 0$, the Wronskian is nonzero. Therefore the computation of the Wronskian is consistent with Abel's theorem.

(b)(15 points) Using the method of variation of parameters, find a particular solution of the inhomogeneous ODE,

$$Ly = t^2 e^{2t}.$$

Solution By the method of variation of parameters, a particular solution is

$$y_d(t) = \int_{t_0}^{t} K(s,t) f(s) ds,$$

where $K(s,t)$ is the *Green's kernel*,

$$K(s,t) = \frac{1}{W(s)}(y_1(s)y_2(t) - y_1(t)y_2(s)).$$

In this case,

$$y_1(s)y_2(t) - y_1(t)y_2(s) = e^{2s}(2t^2 + 2t + 1) - e^{2t}(2s^2 + 2s + 1).$$

Multiplying the Green's kernel by $s^2 e^{2s}$ gives,

$$K(s,t)f(s) = -\frac{1}{4}(e^{2s}(2t^2 + 2t + 1) - e^{2t}(2s^2 + 2s + 1)).$$

Antidifferentiating gives the particular solution,

$$y_d(t) = -\frac{1}{4}\left(\frac{1}{2}(e^{2t} - 1)(2t^2 + 2t + 1) - e^{2t}\left(\frac{2}{3}t^3 + t^2 + t\right)\right).$$

Simplifying, this gives,

$$y_d(t) = \frac{1}{6}t^3 e^{2t} - \frac{1}{8}te^{2t} + \frac{1}{8}(2t^2 + 2t + 1).$$

Of course the last term is a solution of the homogeneous equation. So a particular solution is,

$$y_d(t) = \frac{1}{24}(4t^3 - 3)e^{2t}.$$

Name: ____ **Problem 4**: ________________ /25

Problem 4(25 points) One solution of the linear ODE,

$$\begin{array}{ccccccc} y'' & + & a(t)y' & + & b(t)y & = & \\ y'' & + & \left(-2-\frac{2}{t}\right)y' & + & \frac{4}{t}y & = & 0, \end{array}$$

is the equation $y_1(t) = e^{2t}$.

(a)(10 points) Let $y_2(t)$ be a second solution. Define $W(t)$ to be the Wronskian of $y_1(t)$, $y_2(t)$. This satisfies the differential equation $W' = -a(t)W$. Find a solution $W(t)$ of this differential equation.

Solution The differential equation is separable,

$$\frac{dW}{W} = \left(2 + \frac{2}{t}\right) dt.$$

So,

$$\ln(W) = 2t + 2\ln(t) + C,$$

or equivalently,

$$W(t) = At^2e^{2t}.$$

Taking $A = -4$ gives,

$$W(t) = -4t^2e^{2t}.$$

(b)(15 points) For your solution $W(t)$, solve the first-order linear ODE,

$$y_1(t)v' - y_1'(t)v = W(t).$$

Plug in $y_2(t) = v$ to find a basic solution pair of the ODE, $Ly = 0$.

Solution The ODE is,

$$e^{2t}v' - 2e^{2t}v = -4t^2e^{2t}.$$

Simplifying, this is,

$$v' - 2v = -4t^2.$$

Because 0 is not a root of the characteristic polynomial, by the method of undetermined coefficients we guess,

$$v(t) = a_2t^2 + a_1t + a_0.$$

Substituting in gives,

$$-2a_2t^2 + (2a_2 - 2a_1)t^2 + (a_1 - 2a_0) = -4t^2.$$

Therefore $a_2 = 2$, $a_1 = a_2 = 2$, and $a_0 = a_1/2 = 1$. So the solution is,

$$v(t) = 2t^2 + 2t + 1.$$

So, by Wronskian reduction of order, a basic solution pair of the ODE is,

$$y_1(t) = e^{2t}, \quad y_2(t) = 2t^2 + 2t + 1.$$

This is the same solution pair as in the previous problem! In particular, now that we know the operator Ly, it is straightforward to check that the particular solution $y_d(t)$ of $Ly = t^2e^{2t}$ actually does give a solution.