18.034 SOLUTIONS TO PROBLEM SET 7

Due date: Friday, April 16 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(1)(5 points) p. 439, Problem 6

Solution: To answer all such questions, the simplest method is to apply Gaussian elimination to the augmented matrix,

$$\widetilde{A}_1 = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ 1 & 3 & -4 & 0 & 0 & 1 \end{array}\right]$$

Taking $R_2' = R_2 + R_1, R_3' = R_3 - R_1$, i.e., subtracting the first row from the second row and adding the first row to the third row, gives,

$$\widetilde{A}_2 = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 4 & -4 & -1 & 0 & 1 \end{array}\right]$$

Next, taking $R_2' = (1/2)R_3, R_3' = R_3 - 2R_2$,

$$\widetilde{A}_3 = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{array}\right]$$

Finally, $R_1' = R_1 + R_2$,

$$\widetilde{A}_4 = \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{array}\right]$$

Define A_4 to be the matrix formed by the first 3 columns of $\widetilde{A}_4$. First of all, the nullspace of A equals the nullspace of A_4. Visibly, the nullspace of A_4 is spanned by $[1,1,1]^{\mathrm{T}}$. Therefore the nullspace of A is spanned by $[1,1,1]^{\mathrm{T}}$. Also the column space of A_4 is spanned by the same columns as the column space of A, i.e., the first 2 columns. So the column space of A is spanned by $[1,-1,1]^{\mathrm{T}}$ and $[-1,3,3]^{\mathrm{T}}$.

(a) Therefore $Ax = [1,-1,1]^{\mathrm{T}}$ does have a solution, and the general solution is,

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C \in \mathbb{R}.$$

(b) The general solution of $Ax = 0$ is,

$$x = C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C \in \mathbb{R}.$$

(c) The system $Ax = y$ has a solution iff y is in the column space, i.e., iff

$$y = C_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix},$$

iff $-3y_1 - 2y_2 + y_3 = 0$.

(2)(5 points) p. 439, Problem 12

Solution: We use cofactor expansion along the third row,

$$\det(A) = -2\det(B_1), \quad B_1 = \begin{bmatrix} 5 & 7 & 2 \\ 4 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

To compute the determinant of this matrix, we use row operations, keeping track of how the determinant changes. First take $R_3 = R_1, R_1 = R_3$,

$$\det(B_1) = -\det(B_2), \quad B_2 = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 1 & 1 \\ 5 & 7 & 2 \end{bmatrix}.$$

Next, $R_2' = R_2 - 4R_1$, $R_3' = R_3 - 5R_1$,

$$\det(B_2) = \det(B_3), \quad B_3 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -11 & -15 \\ 0 & -8 & -18 \end{bmatrix}.$$

Next, $R_2' = R_2 - R_3$,

$$\det(B_3) = \det(B_4), \quad B_4 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -3 & 3 \\ 0 & -8 & -18 \end{bmatrix}.$$

Next, $R_3' = R_3 - 2R_2$,

$$\det(B_4) = \det(B_5), \quad B_5 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -3 & 3 \\ 0 & -2 & -24 \end{bmatrix}.$$

Next, $R_2 = R_2' - 2R_3$,

$$\det(B_5) = \det(B_6), \quad B_6 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 51 \\ 0 & -2 & -24 \end{bmatrix}.$$

Finally, $R_3 = R_3 + 2R_2$,

$$\det(B_6) = \det(B_7), \quad B_7 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 51 \\ 0 & 0 & 78 \end{bmatrix}.$$

Because this is an upper triangular matrix, $\det(B_7) = 1 \cdot 1 \cdot 78 = 78$. Therefore $\det(B_2) = \cdots = \det(B_7) = 78$. So $\det(B_1) = -78$. So,

$$\det(A) = (-2)(-78) = 156.$$

(3)(5 points) p. 439, Problem 16

Solution: As shown above, the nullspace has dimension 1. And the column space has dimension 2. Therefore $R(L) + N(L) = 2 + 1$, which is 3 as required.

(4)(5 points) p. 362, Problem 13

Solution: The characteristic polynomial is the determinant of the matrix,

$$\lambda I_{3\times 3} - A = \begin{bmatrix} \lambda+1 & -36 & -100 \\ 0 & \lambda+1 & -27 \\ 0 & 0 & \lambda-5 \end{bmatrix}.$$

Because this is an upper triangular matrix, the determinant is simply $(\lambda+1)(\lambda+1)(\lambda-5) = (\lambda+1)^2(\lambda-5)$. Therefore $\lambda_1 = -1$ is an eigenvalue with multiplicity 2, and $\lambda_2 = 5$ is an eigenvalue with multiplicity 1.

For $\lambda_1 = -1$, the eigenspace is the nullspace of the matrix,

$$-I - A = \begin{bmatrix} 0 & -36 & -100 \\ 0 & 0 & -27 \\ 0 & 0 & -6 \end{bmatrix}.$$

Visibly, the only nullvector is,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

(since the last 2 columns are linearly independent, the rank nullity theorem says the nullspace has dimension 1). Since the multiplicity is 2 but the eigenspace has dimension 1, the eigenspace for $\lambda = -1$ is deficient.

For $\lambda_2 = 5$, the eigenspace is the nullspace of the matrix,

$$5I - A = \begin{bmatrix} 6 & -36 & -100 \\ 0 & 6 & -27 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that $6[-100,-27]^T + 27[-36,6]^T + 262[6,0]^T = [0,0]^T$. Therefore the eigenspace for $\lambda = -5$ is spanned by,

$$v_2 = \begin{bmatrix} 262 \\ 27 \\ 6 \end{bmatrix}.$$

(5)(5 points) p. 362, Problem 18

This can be proved rigorously by induction, but essentially it is obvious by inspection.

(6)(5 points) p. 362, Problem 19

Solution: Let v be an eigenvector for λ, i.e. $Av = \lambda v$. Then, by induction, for every integer $k \geq 1$, $A^k(v) = \lambda^k v$. Therefore v is an eigenvector for A^k with eigenvalue λ^k. This proves the first part.

The second part is a bit vague – it is not stated over what field we are working or which root of μ we are taking. Suppose that λ^k is an eigenvalue of A^k. It is not necessarily true that λ is an eigenvalue of A. For instance, let $A = I_{n\times n}$. Then $(-1)^2 = 1$ is an eigenvalue of A^2. But -1 is not an eigenvalue of A.

Another reasonable interpretation of this part of the problem is this: For every eigenvalue μ of A^k, does there exist an eigenvalue λ of A such that $\mu = \lambda^k$? The answer to this is "yes", at least over $\mathbb{C}$. Let $\lambda_1, \ldots, \lambda_r$ be the distinct complex eigenvalues of A. Then the generalized eigenspace $V^{\text{gen}}_{\lambda_1}, \ldots, V^{\text{gen}}_{\lambda_r}$ give a direct sum decomposition of V. Denote by A_i the restriction of A to $V^{\text{gen}}_{\lambda_i}$. Then $A_i = \lambda_i I + N$, where N is a nilpotent matrix. Therefore, A_i^k is $\lambda_i^k I + BN$ where B is the matrix,

$$B = k\lambda_i^{k-1} I + k(k-1)/2\lambda_i^{k-2} N + \cdots + k\lambda_i N^{k-2} + N^{k-1}.$$

In particular, B commutes with N. So if $N^e = 0$, then $(BN)^e = B^e N^e = B^e 0 = 0$. It follows that $(A_i^k - \lambda_i^k I)^e = 0$. Therefore $V_{\lambda_i}^{\text{gen}}$ is contained in the generalized eigenspace of A^k for the eigenvalue λ_i^k. Because every vector in V can be written as a sum of vectors in $V_{\lambda_1}^{\text{gen}}, \dots, V_{\lambda_r}^{\text{gen}}$, the generalized eigenspaces for $\lambda_1^k, \dots, \lambda_r^k$ already give a direct sum decomposition of V. By our basic theorem that *all* of the nontrivial generalized eigenspaces for A^k give a direct sum decomposition of V, it follows that the *only* eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_r^k$. Therefore every eigenvalue of A^k is the k^{th} power of an eigenvalue of A.

(7)(5 points) p. 362, Problem 21

Solution: This follows immediately by expanding and comparing to Formula (12) on p. 357.

(8)(5 points) p. 362, Problem 22

Solution: Part (a) follows immediately by expanding and comparing to Formula (12). For Part (b), notice that the trance is -5. Since this is a negative real number, the sum of the real parts of the eigenvalues is negative. Therefore, at least one of the eigenvalues has a negative real part.

(9)(5 points) p. 362, Problem 23

Solution: The matrix A is nonsingular iff it has only the trivial nullvector. A nontrivial nullvector is the same thing as an eigenvector for the eigenvalue $\lambda = 0$. Therefore A is nonsingular iff 0 is not an eigenvalue of A.

(10)(5 points) p. 363, Problem 25

Solution: The trace is 4 and the determinant is 4. Therefore the characteristic polynomial is $p_A(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Therefore $\lambda = 2$ is an eigenvalue with multiplicity 2. Because the matrix is not diagonal, clearly the eigenspace is deficient. Choose a random vector, say

$$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then,

$$v_1 = (2I - A)v = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix},$$

is an eigenvector. Therefore a generalized eigenbasis consists of (v_1, v_2). With respect to this ordered basis, the matrix is,

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$