18.03 Recitation 25, May 11, 2010

Autonomous systems

Write x for the population of bugs (in some convenient units), and y for the population of birds. Birds eat bugs, and the two together satisfy the nonlinear autonomous system

$$\begin{cases} \dot{x} &= (2-x-y)x \\ \dot{y} &= (x-1)y \end{cases}$$

(so that in the absence of birds, the bug population grows logistically, and in the absence of bugs, the birds die out exponentially).

1. Find all the critical points of this system.

In order to have $\dot{x} = x(2-x-y) = 0$ and $\dot{y} = y(x-1) = 0$, either $x = y = 0$, or $y = 2 - x = 0$, or $x = 2 - y = 1$. The critical points are then $(0,0)$, $(2,0)$, and $(1,1)$.

2. Find the linearization at each critical point, and sketch the trajectories near them.

We make a matrix of partial derivatives: $J = \begin{bmatrix} 2-2x-y & -x \\ y & x-1 \end{bmatrix}$, and evaluate it at the critical points. At $(0,0)$, we get $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, which describes a saddle with eigenlines parallel to the x and y axes. At $(2,0)$, we get $\begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$, which describes a saddle with the -2 eigenline having slope 0, and the 1 eigenline having slope $-3/2$. At $(1,1)$, we get $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, whose characteristic polynomial is $\lambda^2 + \lambda + 1$. This describes a stable counterclockwise spiral.

3. Locate the phase lines of the bugs-only equation and of the birds-only equation on the (bugs,birds) plane. They form part of the phase portrait of the (bugs,birds) system. Use this and the work from **(2)** to sketch the phase portrait of the (bugs,birds) system.

The bugs-only equation comes from setting $y = 0$, so $\dot{x} = x(2-x)$. This has an unstable critical point at zero, and a stable critical point at 2.

The birds-only equation comes from setting $x = 0$, so $\dot{y} = -y$. This has a stable critical point at zero.

4. Pick some positive initial values of $x(t)$ and $y(t)$ and use your phase portrait to sketch the graphs of $x(t)$ and $y(t)$. As t gets large, x and y converge to limiting values, exponentially. At what exponential rate? Do they overshoot/undershoot the limiting value, and oscillate around it? If so, what it the pseudoperiod of the oscillation?

If $(x(t), y(t))$ starts in the first quadrant, as t gets large, the trajectory will converge to the stable critical point (1,1). The work from (2) shows that this is a counterclockwise spiral and the eigenvalues corresponding to this linearization are $\lambda = \frac{-1\pm\sqrt{3}i}{2}$. So the convergence rate is $e^{-\frac{t}{2}}$, and the trajectory will oscillate around (1,1) with a circular frequency $\sqrt{3}/2$.

5. Now malathion is introduced in an attempt to reduce the bug population. This reduces the rate of reproduction of both species, so the new system is given by

$$\begin{cases} \dot{x} &= (2 - x - y - a)x \\ \dot{y} &= (x - 1 - b)y \end{cases}$$

for certain small positive constants a, b. What happens to the critical point for which both x and y are positive? Is this measure successful in reducing the bug population?

We solve the equations $2 - x - y - a = 0$ and $x - 1 - b = 0$ to get $x = b + 1$ and $y = 2 - x - a = 1 - a - b$. The equilibrium bug population increases by b, and the bird population decreases by $a + b$. The measure is unsuccessful.

18.03 Differential Equations
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