18.03 Recitation 17, April 8, 2010

Convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

1. (a) (a) Compute $t * u(t)$. More generally, compute $q(t) * u(t)$ in terms of $q(t)$.

$t*u(t) = \int_0^t (t-\tau)u(\tau)d\tau$. Note that for any τ in between 0 and t, $u(\tau) = 1$, so $t*u(t) = \int_0^t (t-\tau)d\tau = t^2/2$. More generally, $q(t)*u(t) = \int_0^t q(t-\tau)u(\tau)d\tau = \int_0^t q(t-\tau)d\tau$.

(b) Compute $u(t) * t$. More generally, compute $u(t) * q(t)$ in terms of $q(t)$.

Again, for any τ in between 0 and t, $t-\tau$ is also in between 0 and t, therefore, $u(t) * t = \int_0^t u(t-\tau)\tau d\tau = \int_0^t \tau d\tau = t^2/2$. In general, $u(t) * q(t) = \int_0^t u(t-\tau)q(\tau)d\tau = \int_0^t q(\tau)d\tau$. With the change of variable, i.e., $t-\tau \to \tau$, we see $u(t) * q(t) = \int_0^t q(\tau)d\tau = \int_0^t q(t-\tau)d\tau = q(t) * u(t)$.

What we see here is $t * u(t) = u(t) * t$ and $q(t) * u(t) = u(t) * q(t)$. In fact, the convolution operation "$*$" is commutative. Namely, for any $f(t)$ and $g(t)$, $f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = g * f(t)$.

2.What is the differential operator $p(D)$ whose unit impulse response is the unit step function $u(t)$? In **1(b)** you have computed $u(t)*q(t)$. Is the Assertion true in this case?

We are looking for the differential operator $P(D)$ such that $P(D)u = \delta$. But $Du = \delta$, so $p(D) = D$. The Assertion in this case becomes $x(t) = u(t) * q(t) = \int_0^t q(\tau)d\tau$ is the solution of $Dx = q(t)$ with the rest initial condition. To verify this, we take the derivative of $x(t)$, which leads to $\dot{x}(t) = \frac{d}{dt}(\int_0^t q(\tau)d\tau) = q(t)$, and moreover, $x(0+) = 0$. Therefore, $x(t)$ is indeed the solution of $Dx = q(t)$ and it satisfies the rest initial condition.

3. (a) Assume that $f(t)$ is continuous at $t = a$. What meaning should we give to the product $f(t)\delta(t-a)$?

Consider that $\delta(t-a)$ is zero everywhere except for $t = a$, so the product $f(t)\delta(t-a)$ is zero everywhere except for $t = a$. But at $t = a$, $f(t)$ is simply $f(a)$. Therefore, $f(t)\delta(t-a)$ is again, a delta function at a of "size" $f(a)$. Namely, $f(t)\delta(t-a) = f(a)\delta(t-a)$.

(b) Assume $f(t)$ is continuous, and $f(t)$ vanishes for $t < 0$. Explain why $f(t) * \delta(t-a) = f(t-a)$ for $a \geq 0$.

$f(t) * \delta(t-a) = \int_0^t f(t-\tau)\delta(\tau-a)d\tau$. As we just saw in 3(a), the product $f(t-\tau)\delta(\tau-a) = f(t-a)\delta(\tau-a)$, so the integral becomes $\int_0^t f(t-\tau)\delta(\tau -$

$a)d\tau = \int_0^t f(t-a)\delta(\tau-a)d\tau = f(t-a)\int_0^t \delta(\tau-a)d\tau$. Since $\delta(\tau-a) = u'(\tau-a)$, then when $a \geq 0$, $\int_0^t \delta(\tau-a)d\tau = u(t-a) - u(-a) = 1$. So we have $f(t) * \delta(t-a) = f(t-a)$. When $a = 0$, $f * \delta(t) = f(t)$, so the convolution of f with $\delta(t)$ gets back to f.

4. (a) Verify that $u(t)\frac{1}{\omega_n}\sin(\omega_n t)$ is the unit impulse response for $D^2 + \omega_n^2 I$.

Denote $x(t) = u(t)\frac{1}{\omega_n}\sin(\omega_n t)$. Clearly, when $t < 0$, $x(t) = 0$; when $t > 0$, $x(t) = \frac{1}{\omega_n}\sin(\omega_n t)$, and $Dx = \cos(\omega_n t)$, $D^2 x = -\omega_n \sin(\omega_n t)$. So $(D^2 + \omega_n^2 I)x = 0$ when $t > 0$. Moreover, it's easy to verify that $x(0+) = 0$ and $\dot{x}(0+) = 1$. So $x(t)$ has the right initial condition at $t = 0$, so it's the unit impulse response for $D^2 + \omega_n^2 I$.

(b) Find the solution to $\ddot{x} + x = \sin t$ with initial condition $x(0) = \dot{x}(0) = 0$, using the ERF/resonance.

The complex replacement of the equation is $\ddot{z} + z = e^{it}$, and by Resonant ERF, it has the particular solution $z_p = \frac{te^{it}}{2i}$. So the original equation has a particular solution $x_p = Im(z_p) = -\frac{t}{2}\cos t$, and hence the general solution is $x = -\frac{t}{2}\cos t + c_1 \cos t + c_2 \sin t$. $x(0) = c_1 = 0$, and $\dot{x}(0) = -\frac{1}{2} + c_2 = 0$. So $c_1 = 0$ and $c_2 = \frac{1}{2}$, and the solution is $x = -\frac{t}{2}\cos t + \frac{1}{2}\sin t$.

(c) Compute $\sin t * \sin t$ at $t = 2\pi n$, where n is a positive integer. (Reminder: $\sin^2 t = \frac{1-\cos(2t)}{2}$.)

By the Assertion, $\sin t * \sin t$ should be the solution found in **(b)**. Is the value at $t = 2\pi n$ correct?

$\sin t * \sin t$ at $2\pi n$ is given by $\int_0^{2\pi n} \sin(2\pi n - \tau)\sin\tau d\tau = -\int_0^{2\pi n} \sin^2 \tau d\tau = \int_0^{2\pi n} \frac{\cos(2\tau)-1}{2} d\tau$. $\int_0^{2\pi n} \frac{\cos(2\tau)}{2} d\tau = \frac{1}{4}(\sin(4\pi n) - \sin 0) = 0$, so the integral is simply $-\frac{1}{2} \cdot 2\pi n = -\pi n$. Therefore, $\sin t * \sin t$ at $2\pi n$ should be $-\pi n$. But $x(t)$ from (b) at $2\pi n$ takes value $-\frac{2\pi n}{2}\cos(2\pi n) + \frac{1}{2}\sin(2\pi n) = -\pi n$, so it matches what we found here.