Recitation 1, February 2, 2010

Natural growth and decay, review of logarithm

Solution suggestions

1. Write down a model for the oryx population.

To fix symbols, let's say at time t, the oryx population is $x(t)$. Since the harvesting rate is a *oryx(es) per year*, let's assume t has unit *year(s)* and $x(t)$ has unit *oryx(es)*. The natural growth rate is k, meaning that after some short time Δt year(s) passes, we expect $k\,x(t)\,\Delta t$ new oryxes to appear. However, meanwhile the population is reduced by $a\Delta t$ oryxes due to the harvesting. Therefore, we are led to

$$x(t + \Delta t) \simeq x(t) + k\,x(t)\,\Delta t - a\Delta t,$$

and the unit on both sides is oryx(es). If we let Δt approach 0, then we get the differential equation

$$\frac{dx}{dt} = kx - a.$$

2. Suppose $a = 0$, what is the *doubling time*?

When $a = 0$, the differential equation becomes $\dot{x} = kx$. To solve it, cross multiply to get

$$\frac{dx}{x} = k\,dt,$$

and then integrate both sides. We will get that $\ln|x(t)| = kt + c$, where c is a constant, and hence $x(t) = \pm e^c e^{kt}$. Taking into account the "missing solution" $x \equiv 0$, we can summarize that the general solution to $\dot{x} = kx$ is given by $x(t) = Ce^{kt}$, where C is a constant. In fact, C is the "initial value" $x(0)$. Therefore, the *doubling time* is given by the time T when $x(T) = 2C$, which is equivalent to $e^{kT} = 2$ if C is not 0. Hence,

$$T = \frac{\ln 2}{k}.$$

3. Find the general solution of this equation.

If $k = 0$, then we have $\dot{x} = -a$, with solution $x(t) = -at + C$, with initial value C.

If $k \neq 0$, we do a variable substitution first. Let $u = x - a/k$, so $\dot{u} = \dot{x}$. Then the equation becomes $\dot{u} = ku$, which has solutions $u(t) = Ce^{kt}$ as we discussed in the previous question. Substituting back, we find that $x = u + a/k$, or

$$x(t) = Ce^{kt} + \frac{a}{k}.$$

The initial value is $x(0) = C + a/k$.

4. Check that the proposed solution satisfies the ODE.

For $k = 0$, we have $\dot{x}(t) = -a = kx - a$.

For $k \neq 0$, we have $\dot{x}(t) = Cke^{kt} = k(Ce^{kt} + a/k) - a = kx - a$.

5. There is a "steady state" (also known as constant, or equilibrium) solution. Find it. Does the way the solution depends upon k and a make sense? (That is: do the units come out right? Does it behave right when a is large, or small, and when k is large, or small?)

The constant solution is the one for which $\dot{x} \equiv 0$. If $k = 0$, this only exists in the absence of harvesting, i.e., when $a = 0$, but then all solutions are steady state. If $k \neq 0$, we have to solve: $Cke^{kt} \equiv 0$, so $C = 0$. This means $x(t) = a/k$ for all time t. In other words, $x \equiv a/k$ is the "steady state", which has unit *oryx(es)* on both sides. Furthermore, this is equivalent to $kx \equiv a$. The population is increasing by kx oryx(es) per year because of the natural growth, and decreasing by a oryx(es) per year due to the hunting. Intuitively, when the "increasing rate" is equal to the "decreasing rate", the population is steady. Therefore, the way the steady solution depends on k and a makes sense.

6. Graph the steady state solution and some others, and comment on what they signify. The equilibrium is "unstable." For initial values less than the equlibrium, the solutions stop having meanings in terms of this problem when they become negative. (Of course, this is true for all initial values, but for somewhat different reasons.) In that case, predict the time t_0 at which oryxes will be extirpated. For example, suppose that x_0 is half the steady state: what is t_0 (in terms of k)?

For initial oryx populations above a/k, the harvesting does not cancel out the growth, and the popluation increases without bound. For initial populations below a/k, the harvesting overtakes the growth rate, and the population goes to zero. If we let x_0 denote the initial population, then we can find the extinction time using the following equations: $x_0 = x(0) = Ce^{0k} + a/k = C + a/k$. For $x_0 < a/k$, we have $C = x_0 - a/k < 0$. Solving

$$x(t_0) = (x_0 - a/k)e^{kt_0} + a/k = 0$$

for t_0, we have

$$t_0 = \frac{1}{k} \ln \frac{a}{a - kx_0}.$$

When $x_0 = \frac{a}{2k}$, t_0 is given by $\frac{\ln 2}{k}$.

The following are the graphs of $x(t)$ when $k = 2$, $a = 2000$ and $C = -1000, -500, 0, 500, 1000$ respectively.

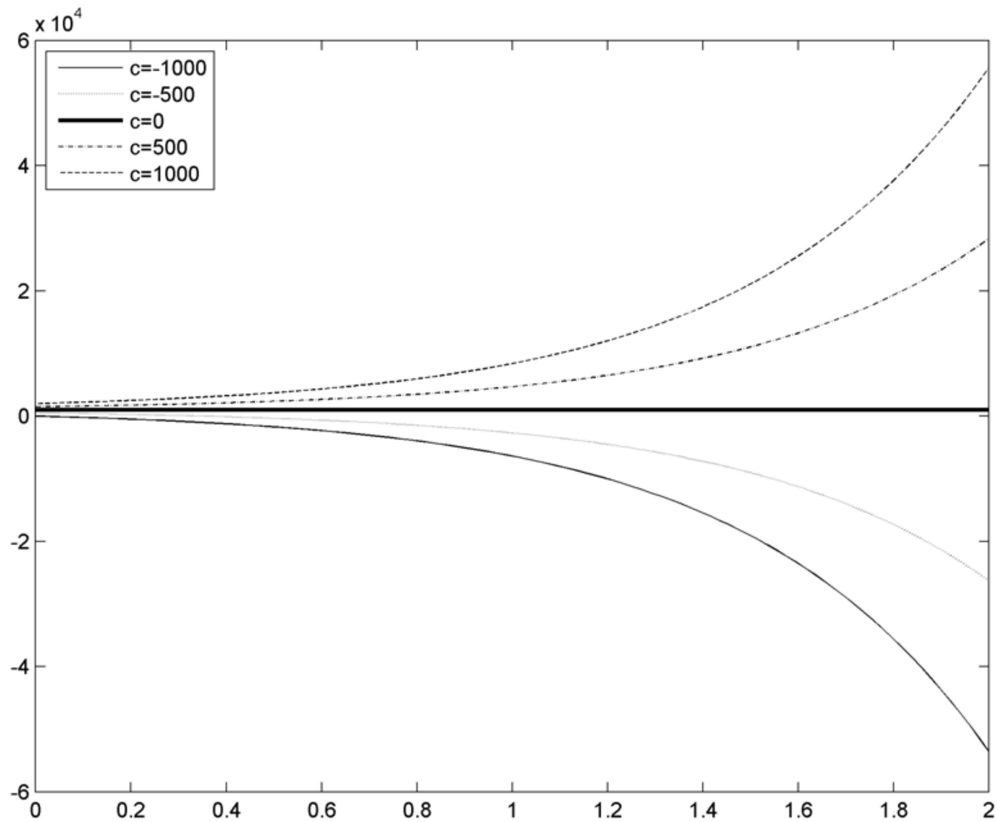

Figure 1: $x\left(t\right)=Ce^{2t}+1000$