S. Stability

1. The notion of stability.

A system is called *stable* if its long-term behavior does not depend significantly on the initial conditions.

It is an important result of mechanics that any system of masses connected by springs (damped or undamped) is a stable system. In network theory, there is a similar result: any RLC-network gives a stable system. In these notes, we investigate for the simplest such systems why this is so.

In terms of differential equations, the simplest spring-mass system or RLC-circuit is represented by an ODE of the form

$$(1) \qquad a_0 y'' + a_1 y' + a_2 y = r(t), \qquad a_i \text{ constants}, \quad t = \text{time}.$$

For the spring-mass system, y is the displacement from equilibrium position, and $r(t)$ is the externally applied force.

For the RLC-circuit, y represents the charge on the capacitor, and $r(t)$ is the electromotive force $\mathcal{E}(t)$ applied to the circuit (or else y is the current and $r(t) = \mathcal{E}'$).

By the theory of inhomogeneous equations, the general solution to (1) has the form

$$(2) \qquad y = c_1 y_1 + c_2 y_2 + y_p, \qquad c_1, c_2 \text{ arbitrary constants},$$

where y_p is a *particular solution* to (1), and $c_1 y_1 + c_2 y_2$ is the *complementary function*, i.e., the general solution to the associated homogeneous equation (the one having $r(t) = 0$).

The initial conditions determine the exact values of c_1 and c_2. So from (2),

$$(3) \qquad \begin{array}{ccc} \textit{the system modeled} & \Longleftrightarrow & \textit{for every choice of } c_1, c_2, \\ \textit{by (1) is stable} & & c_1 y_1 + c_2 y_2 \to 0 \textit{ as } t \to \infty. \end{array}$$

Often one applies the term *stable* to the ODE (1) itself, as well as to the system it models. We shall do this here.

If the ODE (1) is stable, the two parts of the solution (2) are named:

$$(4) \qquad y_p = \textit{steady-state solution} \qquad c_1 y_1 + c_2 y_2 = \textit{transient};$$

the whole solution $y(t)$ and the right side $r(t)$ of (1) are described by the terms

$$y(t) = \textit{output} \qquad\qquad r(t) = \textit{input}.$$

From this point of view, the driving force (or EMF) is viewed as the input to the system, and the response of the system is thought of as the output. So what (2) and (4) are saying is that the output is the sum of two terms: a transient term, which depends on the initial conditions, but whose effects disappear over time, and a steady-state term, which represents more and more closely the response of the system as time goes to ∞.

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2. Conditions for stability: second-order equations.

We now ask under what circumstances the ODE (1) will be stable. In view of the definition, together with (2) and (3), we see that stability concerns just the behavior of the solutions to the associated homogeneous equation

$$(5) \qquad a_0 y'' + a_1 y' + a_2 y = 0 ;$$

the forcing term $r(t)$ plays no role in deciding whether or not (1) is stable.

There are three cases to be considered in studying the stability of (5); they are summarized in the table below, and based on the roots of the characteristic equation

$$(6) \qquad a_0 r^2 + a_1 r + a_2 = 0 .$$

roots	solution to ODE	condition for stability
$r_1 \neq r_2$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 < 0, r_2 < 0$
$r_1 = r_2$	$e^{r_1 t}(c_1 + c_2 t)$	$r_1 < 0$
$a \pm ib$	$e^{at}(c_1 \cos bt + c_2 \sin bt)$	$a < 0$

The first two columns of the table should be familiar, from your work in solving the linear second-order equation (5) with constant coefficients. Let us consider the third column, therefore. In each case, we want to show that if the condition given in the third column holds, then the criterion (3) for stability will be satisfied.

Consider the first case. If $r_1 < 0$ and $r_2 < 0$, then it is immediate that the solution given tends to 0 as $t \to \infty$.

On the other hand, if say $r_1 \geq 0$, then the solution $e^{r_1 t}$ tends to ∞ (or to 1 if $r_1 = 0$). This shows the ODE (5) is not stable, since not all solutions tend to 0 as $t \to \infty$.

In the second case, the reasoning is the same, except that here we are using the limit

$$\lim_{t \to \infty} t e^{rt} = 0 \quad \Leftrightarrow \quad r < 0$$

For the third case, the relevant limit is (there is a similar one involving $\sin t$)

$$\lim_{t \to \infty} e^{at} \cos bt = 0 \quad \Leftrightarrow \quad a < 0 .$$

The three cases can be summarized conveniently by one statement:

Stability criterion for second-order ODE's — root form

$$(7) \qquad a_0 y'' + a_1 y' + a_2 y = r(t) \;\; \textit{is stable} \quad \Leftrightarrow \quad \begin{array}{l} \textit{all roots of } a_0 r^2 + a_1 r + a_2 = 0 \\ \textit{have negative real part.} \end{array}$$

Alternatively, one can phrase the criterion in terms of the coefficients of the ODE; this is convenient, since it doesn't require you to calculate the roots of the characteristic equation.

Stability criterion for second order ODE's — coefficient form. *Assume $a_0 > 0$.*

$$(8) \qquad a_0 y'' + a_1 y' + a_2 y = r(t) \;\; \textit{is stable} \quad \Longleftrightarrow \quad a_0, a_1, a_2 > 0 .$$

The proof is left for the exercises; it is based on the quadratic formula.

3. Stability of higher-order ODE's.

The stability criterion in the root form (7) also applies to higher-order ODE's with constant coefficients:

(9) $(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n)\, y \;=\; f(t)$.

These model more complicated spring-mass systems and multi-loop RLC circuits. The characteristic equation of the associated homogeneous equation is

(10) $a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1}r + a_n \;=\; 0$.

The real and complex roots of the characteristic equation give rise to solutions to the associated homogeneous equation just as they do for second order equations. (For a k-fold repeated root, one gets additional solutions by multiplying by $1, t, t^2, \ldots t^{k-1}$.)

The reasoning which led to the above stability criterion for second-order equations applies to higher-order equations just as well. The end result is the same:

Stability criterion for higher-order ODE's — root form

(11)   *ODE (9) is stable $\iff$ all roots of (10) have negative real parts;*

that is, all the real roots are negative, and all the complex roots have negative real part.

There is a stability criterion for higher-order ODE's which uses just the coefficients of the equation, but it is not so simple as the one (8) for second-order equations. Without loss of generality, we may assume that $a_0 > 0$. Then it is not hard to prove (see the Exercises) that

(12)   *ODE (9) is stable $\Rightarrow$ $a_0, \ldots, a_n > 0$* .

The converse is not true (see the exercises). For an implication $\Leftarrow$, the coefficients must satisfy a more complicated set of inequalities, which we give without proof, known as the

Routh-Hurwitz conditions for stability   *Assume $a_0 > 0$; ODE (9) is stable $\Leftrightarrow$*

in the determinant below, all of the n principal minors (i.e., the subdeterminants in the upper left corner having sizes respectively $1, 2, \ldots, n$) are > 0 when evaluated.

(13)
$$
\begin{vmatrix}
a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\
a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \ldots & \ldots & \ldots & a_n
\end{vmatrix}
$$

In the determinant, we define $a_k = 0$ if $k > n$; thus for example, the last row always has just one non-zero entry, a_n.

Exercises: Section 2G

18.03 Differential Equations
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