I. Impulse Response and Convolution

1. Impulse response. Imagine a mass m at rest on a frictionless track, then given a sharp kick at time $t = 0$. We model the kick as a constant force F applied to the mass over a very short time interval $0 < t < \epsilon$. During the kick the velocity $v(t)$ of the mass rises rapidly from 0 to $v(\epsilon)$; after the kick, it moves with constant velocity $v(\epsilon)$, since no further force is acting on it. We want to express $v(\epsilon)$ in terms of F, ϵ, and m.

By Newton's law, the force F produces a constant acceleration a, and we get

$$(1) \qquad F = ma \quad \Rightarrow \quad v(t) = at, \quad 0 \le t \le \epsilon \quad \Rightarrow \quad v(\epsilon) = a\epsilon = \frac{F\epsilon}{m} \ .$$

If the mass is part of a spring-mass-dashpot system, modeled by the IVP

$$(2) \qquad my'' + cy' + ky = f(t), \qquad y(0) = 0,\ y'(0^-) = 0,$$

to determine the motion $y(t)$ of the mass, we should solve (2), taking the driving force $f(t)$ to be a constant F over the time interval $[0, \epsilon]$ and 0 afterwards. But this will take work and the answer will need interpretation.

Instead, we can both save work and get quick insight by solving the problem approximately, as follows. Assume the time interval ϵ is negligible compared to the other parameters. Then according to (1), the kick should impart the instantaneous velocity $F\epsilon/m$ to the mass, after which its motion $y(t)$ will be the appropriate solution to the homogeneous ODE associated with (2). That is, if the time interval ϵ for the initial kick is very small, the motion is approximately given (for $t \ge 0$) by the solution $y(t)$ to the IVP

$$(3) \qquad my'' + cy' + ky = 0, \qquad y(0) = 0,\ \ y'(0) = \frac{F\epsilon}{m} \ .$$

Instead of worrying about the constants, assume for the moment that $F\epsilon/m = 1$; then the IVP (3) becomes

$$(4) \qquad my'' + cy' + ky = 0, \qquad y(0) = 0,\ \ y'(0) = 1 \ ;$$

its solution for $t > 0$ will be called $w(t)$, and in view of the physical problem, we define $w(t) = 0$ for $t < 0$.

Comparing (3) and (4), we see that we can write the solution to (3) in terms of $w(t)$ as

$$(5) \qquad y(t) = \frac{F\epsilon}{m}\ w(t) \ ,$$

for since the ODE (4) is linear, multiplying the initial values $y(0)$ and $y'(0)$ by the same factor $F\epsilon/m$ multiplies the solution by this factor.

The solution $w(t)$ to (4) is of fundamental importance for the system (2); it is often called in engineering the **weight function** for the ODE in (4). A longer but more expressive name for it is the **unit impulse response** of the system: the quantity $F\epsilon$ is called in physics the *impulse* of the force, as is $F\epsilon/m$ (more properly, the *impulse/unit mass*), so that if $F\epsilon/m = 1$, the function $w(t)$ is the response of the system to a unit impulse at time $t = 0$.

Example 1. Find the unit impulse response to an undamped spring-mass system having (circular) frequency ω_0.

Solution. Taking $m = 1$, the IVP (4) is $y'' + \omega_0^2 y = 0$, $y(0) = 0$, $y'(0) = 1$, so that $y_c = a\cos\omega_0 t + b\sin\omega_0 t$; substituting in the initial conditions, we find

$$w(t) = \begin{cases} \frac{1}{\omega_0}\sin\omega_0 t, & t > 0; \\ 0, & t < 0. \end{cases}$$

Example 2. Find the unit impulse response to a critically damped spring-mass-dashpot system having e^{-pt} in its complementary function.

Solution. Since it is critically damped, it has a repeated characteristic root $-p$, and the complementary function is $y_c = e^{-pt}(c_1 + c_2 t)$. The function in this family satisfying $y(0) = 0$, $y'(0) = 1$ must have $c_1 = 0$; it is $t\,e^{-pt}$, either by differentiation or by observing that its power series expansion starts $t(1 - pt + ...) \approx t$.

2. Superposition.

We now return to the general second-order linear ODE with constant coefficients

$$(6) \qquad my'' + cy' + ky = f(t), \qquad \text{or} \qquad L(y) = f(t), \quad \text{where} \quad L = mD^2 + cD + k.$$

We shall continue to interpret (6) as modeling a spring-mass-dashpot system, this time with an arbitrary driving force $f(t)$.

Since we know how to solve the associated homogeneous ODE, i.e., find the complementary solution y_c, our problem with (6) is to find a particular solution y_p. We can do this if $f(t)$ is "special" — sums of products of polynomials, exponentials, and sines and cosines. If $f(t)$ is periodic, or we are interested in it only on a finite interval, we can try expanding it into a Fourier series over that interval , and obtaining the particular solution y_p as a Fourier series. But what can we do for a general $f(t)$?

We use the *linearity* of the ODE (6), which allows us to make use of a

Superposition principle

If $f(t) = f_1(t) + \ldots + f_n(t)$ and y_i are corresponding particular solutions:

$$L(y_i) = f_i(t), \qquad i = 1, \ldots, n,$$

then $y_p = y_1 + \ldots + y_n$ is a particular solution to (2).

Proof. Using the linearity of the polynomial operator L, the proof takes one line:

$$L(y_p) \;=\; L(y_1) + \ldots + L(y_n) \;=\; f_1(t) + \ldots + f_n(t) \;=\; f(t).$$

Of course a general $f(t)$ is not the sum of a finite number of simpler functions. But over a finite interval we can approximate $f(t)$ by a sum of such functions.

Let the time interval be $0 < t < x$; we want to find the value of the particular solution $y_p(t)$ to (6) at the time $t = x$. We divide the time interval $[0, x]$ into n equal small intervals of length Δt:

$$0 = t_0,\ t_1,\ t_2,\ \ldots,\ t_n = x, \qquad t_{i+1} - t_i = \Delta t.$$

Over the time interval $[t_i, t_{i+1}]$ we have approximately $f(t) \approx f(t_i)$, and therefore if we set

$$f_i(t) = \begin{cases} f(t_i), & t_i \le t < t_{i+1}; \\ 0, & \text{elsewhere,} \end{cases} \qquad i = 0, 1, \ldots, n-1 ,$$

we will have approximately

(7) $$f(t) \approx f_0(t) + \ldots + f_{n-1}(t), \qquad 0 < t < x .$$

We now apply our superposition principle. Since $w(t)$ is the response of the system in (6) to a unit impulse at time $t = 0$, then the response of (6) to the impulse given by $f_i(t)$ (in other words, the particular solution to (6) corresponding to $f_i(t)$) will be

(8) $$f(t_i)w(t - t_i)\Delta t;$$

we translated $w(t)$ to the right by t_i units since the impulse is delivered at time t_i rather than at $t = 0$; we multiplied it by the constant $f(t_i)\Delta t$ since this is the actual impulse: the force $f_i(t)$ has magnitude $f(t_i)$ and is applied over a time interval Δt.

Since (7) breaks up $f(t)$, and (8) gives the response to each $f_i(t)$, the superposition principle tells us that the particular solution is approximated by the sum:

$$y_p(t) = \sum_0^{n-1} f(t_i)w(t - t_i)\Delta t, \qquad 0 \le t \le x,$$

so that at the time $t = x$,

(9) $$y_p(x) \approx \sum_0^{n-1} f(t_i)w(x - t_i)\Delta t .$$

We recognize the sum in (9) as the sum which approximates a definite integral; if we pass to the limit as $n \to \infty$, i.e., as $\Delta t \to 0$, in the limit the sum becomes the definite integral and the approximation becomes an equality:

(10) $$y_p(x) = \int_0^x f(t)\, w(x - t)\, dt \qquad \textbf{system response to } f(t)$$

> In effect, we are imagining the driving force $f(t)$ to be made up of an infinite succession of infinitely close kicks $f_i(t)$; by the superposition principle, the response of the system can then be obtained by adding up (via integration) the responses of the system to each of these kicks.

Which particular solution does (10) give? The answer is:

(11) $$\text{for } y_p \text{ as in (10)}, \qquad y_p(0) = 0, \quad y'(0) = 0.$$

The first equation is clear from (10); we will derive the second in the next section.

The formula (10) is a very remarkable one. It expresses a particular solution to a second-order differential equation directly as a definite integral, whose integrand consists of two parts: a factor $w(x - t)$ depending only on the left-hand-side of (6) — that is, only on the spring-mass-dashpot system itself, not on how it is being driven — and a factor $f(t)$ depending only on the external driving force. For example, this means that once the unit impulse response $w(t)$ is calculated for the system, one only has to put in the different driving forces to determine the responses of the system to each.

The formula (10) makes superposition clear: to the sum of driving forces corresponds the sum of the corresponding particular solutions.

Still another advantage of our formula (10) is that it allows the driving force to have discontinuities, as long as they are isolated, since such functions can always be integrated. For instance, $f(t)$ could be a step function or the square wave function.

Let us check out the formula (10) in some simple cases where we can find the particular solution y_p also by the method of undetermined coefficients.

Example 3. Find the particular solution given by (10) to $y''+y = A$, where $D = d/dx$.

Solution. From Example 1, we have $w(t) = \sin t$. Therefore for $x \geq 0$, we have

$$y_p(x) = \int_0^x A\sin(x-t)\,dt = A\cos(x-t)\Big]_0^x = A(1-\cos x).$$

Here the method of undetermined coefficients would produce $y_p = A$; however, $A - A\cos x$ is also a particular solution, since $-A\cos x$ is in the complementary function y_c. Note that the above y_p satisfies (11), whereas $y_p = A$ does not.

Example 4. Find the particular solution for $x \geq 0$ given by (10) to $y'' + y = f(x)$, where $f(x) = 1$ if $0 \leq x \leq \pi$, and $f(x) = 0$ elsewhere.

Solution. Here the method of Example 3 leads to two cases: $0 \leq x \leq \pi$ and $x \geq \pi$:

$$y_p = \int_0^x f(t)\sin(x-t)\,dt = \begin{cases} \displaystyle\int_0^x \sin(x-t)\,dt = \cos(x-t)\Big]_0^x = 1-\cos x, & 0 \leq x \leq \pi; \\ \displaystyle\int_0^\pi \sin(x-t)\,dt = \cos(x-t)\Big]_0^\pi = -2\cos x, & x \geq \pi \end{cases}$$

3. Leibniz' Formula.

To gain further confidence in our formula (10), which was obtained as a limit of approximations of varying degrees of shadiness, we want to check that it satisfies the ODE (6), with the initial conditions $y(0) = y'(0) = 0$.

To do this, we will have to differentiate the right side of (10) with respect to x. The following theorem tells us how to do this.

Theorem. *If the integrand $g(x,t)$ and its partial derivative $g_x(x,t)$ are continuous, then*

$$\frac{d}{dx}\int_a^b g(x,t)\,dt \;=\; \int_a^b g_x(x,t)\,dt. \tag{12}$$

When we try to apply the theorem to differentiating the integral in (10), a difficulty arises because x occurs not just in the integrand, but as one of the limits as well. The best way to handle this is to give these two x's different names: u and v, and write

$$F(u,v) \;=\; \int_0^u g(v,t)\,dt, \qquad u = x, \quad v = x.$$

Using the 18.02 chain rule $\quad \dfrac{d}{dx}F(u,v) \;=\; \dfrac{\partial F}{\partial u}\dfrac{du}{dx} + \dfrac{\partial F}{\partial v}\dfrac{dv}{dx},\quad$ we get

$$\frac{d}{dx}\int_0^u g(v,t)\,dt \;=\; g(v,u) + \int_0^u \frac{\partial}{\partial v} g(v,t)\,dt\ ,$$

by the Second Fundamental Theorem of calculus and the preceding Theorem; then if we substitute $u = x$ and $v = x$, we get

$$(13) \qquad \frac{d}{dx}\int_0^x g(x,t)\,dt \;=\; g(x,x) + \int_0^x \frac{\partial}{\partial x} g(x,t)\,dt \qquad\qquad \textbf{Leibniz' Formula}.$$

We can now use Leibniz' formula to show that (10) satisfies the ODE (6); we have

$$\begin{aligned} y_p \;&=\; \int_0^x f(t)w(x-t)\,dt \\ y_p' \;&=\; f(x)w(x-x) + \int_0^x f(t)w'(x-t)\,dt; \end{aligned}$$

the first term on the right is 0 since $w(0) = 0$ by (5); using Leibniz' formula once more:

$$y_p'' \;=\; f(x)w'(x-x) + \int_0^x f(t)w''(x-t)\,dt;$$

again, the first term on the right is $f(x)$ since $w'(0) = 1$ by (5); multipying each of the three preceding equations by the appropriate coefficient of the ODE and then adding the three equations gives

$$\begin{aligned} y_p'' + ay_p' + by_p \;&=\; f(x) \;+\; \int_0^x f(t)\left[w''(x-t) + aw'(x-t) + bw(x-t)\right]dt \\ &=\; f(x), \end{aligned}$$

since for any independent variable, $w''(u) + aw'(u) + bw(u) = 0$, and therefore the same is true if u is replaced by $x - t$.

This shows that the integral in (10) satisfies the ODE; as for the initial conditions, we have $y_p(0) = 0$ from the definition of the integral in the equation for y_p above, and $y_p'(0) = 0$ from the equation for y_p' and the fact that $w(0) = 0$.

4. Convolution.

Integrals of the form

$$\int_0^x f(t)w(x-t)\,dt$$

occur widely in applications; they are called "convolutions" and a special symbol is used for them. Since w and f have a special meaning in these notes related to second-order ODE's and their associated spring-mass-dashpot systems, we give the definition of convolution using fresh symbols.

Definition. The **convolution** of $u(x)$ and $v(x)$ is the function of x defined by

$$u * v \;=\; \int_0^x u(t)v(x-t)\,dt. \tag{14}$$

The form of the convolution of two functions is not really predictable from the functions. Two simple and useful ones are worth remembering:

$$e^{ax} * e^{bx} = \frac{e^{ax}-e^{bx}}{a-b}, \quad a \neq b; \qquad\qquad e^{ax} * e^{ax} = x\,e^{ax} \tag{15}$$

Proof. We do the first; the second is similar. If $a \neq b$,

$$e^{ax} * e^{bx} = \int_0^x e^{a(t)}e^{b(x-t)}dt = e^{bx}\int_0^x e^{(a-b)t}dt = e^{bx}\frac{e^{(a-b)t}}{a-b}\Big]_0^x = e^{bx}\frac{e^{(a-b)x}-1}{a-b} = \frac{e^{ax}-e^{bx}}{a-b}.$$

Properties of the convolution.

(*linearity*) $\quad (u_1+u_2)*v = u_1*v + u_2*v, \qquad (cu)*v = c(u*v);$

$\quad u*(v_1+v_2) = u*v_1 + u*v_2, \qquad u*(cv) = c(u*v);$

which follow immediately from the corresponding properties of the definite integral.

(*commutativity*) $\quad u*v = v*u$

Since the definition (14) of convolution does not treat u and v symmetrically, the commutativity is not obvious. We will prove the commutativity later using the Laplace transform. One can also prove it directly, by making a change of variable in the convolution integral. As an example, the formula in (15) shows that $e^{ax}*e^{bx} = e^{bx}*e^{ax}$.

5. Examples of using convolution.

Higher-order linear ODE's. The formula $y_p(x) = f(x)*w(x)$ given in (10) also holds for the n-th order ODE $p(D)y = f(t)$ analogous to (6); the weight function $w(t)$ is defined to be the unique solution to the IVP

$$p(D)y = 0, \qquad w(0) = w'(0) = \ldots w^{(n-2)}(0) = 0, \quad w^{(n-1)}(0) = 1. \tag{16}$$

As in the second-order case, $w(t)$ is the response of the system to the driving force $f(t)$ given by a unit impulse at time $t = 0$.

Example 5. Verify $y_p = f(x)*w(x)$ for the solution to the first-order IVP

$$y' + ay = f(x); \quad y(0) = 0. \tag{17}$$

Solution. According to (16), the weight function $w(t)$ should be the solution of the associated homogeneous equation satisfying $w(0) = 1$; it is therefore $w = e^{-at}$. Using the standard integrating factor e^{ax} to solve (17), and a definite integral to express the solution,

$$\begin{aligned} (y\,e^{ax})' &= f(x)e^{ax} \\ y_p\,e^{ax} &= \int_0^x f(t)e^{at}\,dt \qquad t \text{ is a dummy variable} \\ y_p &= \int_0^x f(t)e^{-a(x-t)}\,dt \\ y_p &= f(x)*e^{-ax}. \end{aligned}$$

Example 6. Radioactive dumping. A radioactive substance decays exponentially:

$$R = R_0 e^{-at}, \tag{18}$$

where R_0 is the initial amount, $R(t)$ the amount at time t, and a the decay constant.

A factory produces this substance as a waste by-product, and it is dumped daily on a waste site. Let $f(t)$ be the rate of dumping; this means that in a relatively small time period $[t_0, t_0 + \Delta t]$, approximately $f(t_0)\Delta t$ grams of the substance is dumped.

The dumping starts at time $t = 0$.

Find a formula for the amount of radioactive waste in the dump site at time x, and express it as a convolution.

Solution. Divide up the time interval $[0, x]$ into n equal intervals of length Δt, using the times

$$t_0 = 0,\ t_1,\ \ldots,\ t_n = x.$$

$$\text{amount dumped in the interval } [t_i, t_{i+1}] \ \approx\ f(t_i)\Delta t\ ;$$

By time x, it will have decayed for approximately the length of time $x - t_i$; therefore, according to (18), at time x the amount of waste coming from what was dumped in the time interval $[t_i, t_{i+1}]$ is approximately

$$f(t_i)\Delta t \cdot e^{-a(x-t_i)};$$

this shows that the

$$\text{total amount at time } x \ \approx\ \sum_0^{n-1} f(t_i) e^{-a(x-t_i)} \Delta t\ .$$

As $n \to \infty$ and $\Delta t \to 0$, the sum approaches the corresponding definite integral and the approximation becomes an equality. So we conclude that

$$\text{total amount at time } x\ =\ \int_0^x f(t) e^{-a(x-t)}\, dt\ =\ f(x) * e^{-ax};$$

i.e., the amount of waste at time x is the convolution of the dumping rate and the decay function.

Example 7. Bank interest. On a savings account, a bank pays the continuous interest rate r, meaning that a sum A_0 deposited at time $t = 0$ will by time t grow to the amount $A_0 e^{rt}$.

Suppose that starting at day $t = 0$ a Harvard square juggler deposits every day his take, with deposit rate $d(t)$ — i.e., over a relatively small time interval $[t_0, t_0 + \Delta t]$, he deposits approximately $d(t_0)\Delta t$ dollars in his account. Assuming that he makes no withdrawals and the interest rate doesn't change, give with reasoning an approximate expression (involving a convolution) for the amount of money in his account at time $t = x$.

Solution. Similar to Example 6, and left as an exercise.

Exercises: Section 2H