18.03 Class 34, April 30, 2010

Complex or repeated eigenvalues

1. Eigenvalues and coefficients
2. Complex eigenvalues
3. Repeated eigenvalues
4. Defective and complete

[1] We were solving $u' = Au$, with $A = [a b ; c d]$:
$A = [-2 1 ; -4 3]$ for example. Or u could be a 3D vector, and A a 3x3 matrix.

(1) Find the eigenvalues = roots λ_1, λ_2 of the characteristic polynomial

$$p_A(\lambda) = \det(A - \lambda I)$$

(2) For each eigenvalue, find a nonzero eigenvector:

$$(A - \lambda_1 I) v_1 = 0 \qquad (A - \lambda_2 I) v_2 = 0$$

Normal modes: $u_1 = e^{\lambda_1 t} v_1$, $u_2 = e^{\lambda_2 t} v_2$.

(3) General solution is linear combination of these: $u = c_1 u_1 + c_2 u_2$.

There are a few problems with this, but before pointing them out let me make three comments:

(a) Any multiple of an eigenvector is another eigenvector for the same eigenvalue; they form a line, an "eigenline."

(b) The zero vector is an eigenvector for every value lambda, whether lambda is an eigenvalue or not. Most of the time 0 is the ONLY eigenvector for value lambda ; lambda is an eigenvalue exactly when there is a *nonzero* eigenvector for that value.

(c) $p_A(\lambda) = \det(A - \lambda I)$

$$= \lambda^2 - (a+d) \lambda - (ad-bc)$$

The sum of the diagonal terms of a square matrix is the "trace" of A , $\operatorname{tr} A$, so

$$p_A(\lambda) = \lambda^2 - (\operatorname{tr} A) \lambda + (\det A)$$

In our example, $\operatorname{tr} A = 1$ and $\det A = -2$, and

$$p_A(\lambda) = \lambda^2 - \lambda - 2 .$$

[2] There may be no ray solutions. Romeo and Juliet provided one example. Or, what about

$$A = [1 2 ; -2 1] .$$

Let's apply the method and see what happens. $\mathrm{tr}(A) = 2$, $\det(A) = 5$, so

$$p_A(\lambda) = \lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 4$$

which has roots $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$.

We could abandon the effort at this point, but we had so much fun and success with complex numbers earlier that it seems we should carry on.

Find an eigenvector for $\lambda_1 = 1 + 2i$:

$$A - (1+2i)I : \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Standard method: use the entries in the top row in reverse order with one sign changed: $[\, 2 \,;\, 2i \,]$ or, easier, in this case,

$$v_1 = [\, 1 \,;\, i \,].$$

This is set up so the top entry in the product is 0 . We have a chance to check our work (mainly the calculation of the eigenvalues) by seeing that the bottom entry in the product is 0 too:

$$-2 \cdot 1 - 2i \cdot i = 0$$

$[\, 1 \,;\, i \,]$ is a vector with complex entries. OK, so be it. It's hard to visualize, perhaps, and doesn't represent a point on the plane, but we can still compute with it just fine.

Since λ_2 = conjugate of λ_1,
an eigenvector for λ_2 is given by the conjugate of v_1:

$$v_2 = [\, 1 \,;\, -i \,]$$

So the normal modes

$$v(t) = e^{(1+2i)t} [\, 1 \,;\, i \,] \quad , \quad \bar{v}(t) = e^{(1-2i)t} [\, 1 \,;\, -i \,] \,.$$

As in the case of second order equations, the real and imaginary parts of solutions are again solutions,

$$u1 = (v + \bar{v})/2 = \mathrm{Re}(v) \quad , \quad u2 = (v - \bar{v})/(2i)$$

and we really only need to write down one of the normal modes.

So these are real solutions:

$$u = e^{(1+2i)t} [\, 1 \,;\, i \,]$$

$$= e^t \left(\cos(2t) + i \sin(2t) \right) \left([1;0] + i[0;1] \right) \quad \text{so}$$

$$u1 = \mathrm{Re}(u) = e^t \left(\cos(2t) [1;0] - \sin(2t) [0;1] \right)$$

$$= e^t [\cos(2t) \,;\, -\sin(2t)] \quad \text{and}$$

$$u2 = \mathrm{Im}(u) = e^t \left(\cos(2t) [0;1] + \sin(2t) [1;0] \right)$$

= e^t [sin(2t) ; cos(2t)]

These are two independent real solutions. Both spiral around the origin, clockwise, while fleeing away from it exponentially. They satisfy

u1(0) = [1;0] , u2(0) = [0;1] .

I showed their trajectories on the Mathlet Linear Phase Portraits: Matrix Entry.

The general real solution is

a u1 + b u2 , a, b real .

It is very hard for me to visualize the fact that all those spirals are linear combinations of any two of them.

Summary: Nonreal eigenvalues lead to spiral solutions.
Positive real parts lead to solutions going to infinity with t ("unstable")
Negative real parts lead to solutions going to zero with t ("stable")
Zero real parts lead to solutions parametrizing ellipses.

So we discover that the possibility of complex eigenvalues really isn't a failure of the method at all. There are in fact ray solutions, but they are complex and don't show up on our real phase plane.

[3] Second problem with our method: Illustrated by

A = [-2 1 ; -1 0]

p_A(lambda) = lambda^2 + 2 lambda + 1 = (lambda + 1)^2

which has only one root, "repeated": lambda_1 = lambda_2 = -1.

Still, find an eigenvector:

A - (-1)I = [-1 1 ; -1 1][? ; ?] = [0 ; 0] : v = [1 ; 1]

or any nonzero multiple. ALL eigenvectors for A lie on the line containing 0 and [1 ; 1]. I showed a picture of the phase portrait, which shows only one pair of opposed ray trajectories.

So there is (up to multiples) only one normal mode:

u_1 = e^{-t} [1 ; 1]

But we need another solution. Here is how to find one; I won't go into details, just give you the method.

Write down the same matrix A - lambda_1 I but now find a vector w such that

(A - lambda1 I) w = v .

Then

u_2 = e^{lambda_1 t} (t v + w)

is a second solution.

In our case:

[-1 1 ; -1 1] [? ; ?] = [1 ; 1]

has solution [0;1] , so

u_2 = e^{-t} (t [1;1] + [0;1])

= e^{-t} [t ; t+1]

[0;1] isn't the only vector that works here; [0;1] + c v does too for any constant c. It doesn't matter which one you pick.

With this choice, u_1(0) = [1;1] , u_2(0) = [0;1] .

The general solution is

u = a u_1 + b u_2 .

To learn more about all this you should take 18.06.

Didn't get to talk about this on Friday:

[4] A matrix with a repeated eigenvalue but only one lineful of eigenvectors is called "defective." A matrix can have a repeated eigenvalue and not be defective:

A = [2 0 ; 0 2]

for example has characteristic polynomial

p(lambda) = lambda^2 - 4 lambda + 4 = (lambda - 2)^2

so lambda_1 = lambda_2 = 2. To find an eigenvector consider

A - lambda_1 I : [0 0 ; 0 0] [? ; ?] = [0 ; 0]

Now ANY vector is an eigenvector! Instead of only one line you get the entire plane. For any vector v ,

e^{2t} v

is a solution, and every solution is a normal mode. This is called the "complete" case.

In the 2x2 case, if the eigenvalue is repeated you are in the defective case unless the matrix is precisely [\lambda_1 , 0 ; 0 , \lambda_1]